Branes and Categorifying Integrable Lattice Models

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Scope of Presentation

- Introduction/Motivation
- Summary of results
- 4d Chern-Simons theory from topological-holomorphic twist of D4-NS5 system
- Categorification of R-matrix elements
- S-dual 4d Chern-Simons theory
- Conclusion and Future Work

Introduction/Motivation

- 4d Chern-Simons theory has the action[1]

$$S = \frac{1}{\hbar}\int_{Y\times\Sigma} C \wedge \mathrm{Tr}\Big(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\Big), \qquad (1)$$

  where $\mathcal{A}$ is a complex-valued gauge field, Y is a framed 2-manifold, and Σ is $\mathbb{C}$, $\mathbb{C}^\times$ or $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ endowed with a meromorphic one-form $C = C(z)dz$ (with no zeros).
- Within the realm of **perturbation theory**, this gauge theory is well-defined, and realizes the **Yang-Baxter equation** with spectral parameter.

1. K. Costello, *Supersymmetric gauge theory and the Yangian*, arXiv:1303.2632
K. Costello, E. Witten, M. Yamazaki, *Gauge Theory and Integrability, I, II*, arXiv:1709.09993, 1802.01579

- Outside of perturbation theory, 4d CS is not well-understood - path integral is **exponentially divergent**.
- Suggestion[2] - **Nonperturbative definition** comes from the **D4-NS5 system** of string theory, similar to how the D3-NS5 system realizes the nonperturbative 3d analytically-continued Chern-Simons theory.[3]
- Such a string realization could also allow us to **categorify** the R-matrix elements/Yang-Baxter equation/Yangian in higher dimensions via **string dualities**.

2. E. Witten, *Integrable Lattice Models From Gauge Theory*, arXiv:1611.00592
3. E. Witten, *Fivebranes and Knots*, Quantum Topology **3** (1) (2012) 1–137

This talk is based on

- M. Ashwinkumar, M.-C. Tan, Q. Zhao, *Branes and Categorifying Integrable Lattice Models*, ATMP (in press), arXiv:1806.02821

Summary of results

- A topological-holomorphic sector of the D4-NS5 system with a meromorphic RR 1-form is equivalent to Costello's 4d Chern-Simons theory

$$\int_{\Gamma} \mathcal{D}\mathcal{A}_{\widetilde{\alpha}0}\mathcal{D}\mathcal{A}_{z0}\mathcal{D}\mathcal{A}_{\bar{z}0}\ e^{\frac{i}{g_5^2}\int_{\partial M} C\wedge \mathrm{Tr}\left(\mathcal{A}_0\wedge d\mathcal{A}_0+\frac{2}{3}\mathcal{A}_0\wedge\mathcal{A}_0\wedge\mathcal{A}_0\right)} \tag{2}$$

with nonperturbative integration cycle, Γ, defined by

$$\begin{aligned} \mathcal{F}_{\widetilde{\alpha}\widetilde{\beta}0} &= 0 \\ \mathcal{F}_{\widetilde{\alpha}\bar{z}0} &= 0 \\ 2i\mathcal{F}_{z\bar{z}0} - D_{\widetilde{\alpha}0}\phi_0^{\widetilde{\alpha}} &= 0 \end{aligned} \tag{3}$$

- Using T- and S-duality, we arrive at the NS5-D5 brane system, the supersymmetric Hilbert space of which is defined by the Floer cohomology of the 6d equations

$$\boxed{\begin{aligned} \mathcal{F}_{ij} &= 0 \\ \mathcal{F}_{i\bar{z}} &= 0 \\ \mathcal{F}_{i\overline{w}} &= 0 \\ F_{\overline{w}\bar{z}} &= 0 \\ 2iF_{z\bar{z}} + 2iF_{w\overline{w}} - D_j\phi^j &= 0 \end{aligned}} \tag{4}$$

which interpolate solutions of the 5d equations

$$\boxed{\begin{aligned} \mathcal{F}_{\beta\gamma} &= 0 \\ \mathcal{F}_{\alpha\bar{z}} &= 0 \\ 2iF_{z\bar{z}} - D_\beta\phi^\beta &= 0 \end{aligned}} \tag{5}$$

- Via the aforementioned dualities, we are able to express each R-matrix element in terms of a trace over this Hilbert space:

$$\boxed{R^{12}_{IK,JL}(z_1, z_2) = \mathrm{Tr}_{\mathcal{H}}\left((-1)^{\mathrm{F}} e^{-\hbar P} W^1_{IJ}(z_1) W^2_{KL}(z_2)\right)} \qquad (6)$$

(thereby categorifying each R-matrix element), and we are also able to categorify the Yang-Baxter equation

$$\sum_{O,P,Q} R^{12}_{NM,QO}(z_1, z_2) R^{13}_{QL,IP}(z_1, z_3) R^{23}_{OP,JK}(z_2, z_3) = \sum_{R,S,T} R^{23}_{ML,RT}(z_2, z_3) R^{13}_{NT,SK}(z_1, z_3) R^{12}_{SR,IJ}(z_1, z_2)$$

as

$$\boxed{\mathrm{Tr}_{\mathcal{H}}\left((-1)^{\mathrm{F}} e^{-\hbar P} W^1_{NI}(z_1) W^2_{MJ}(z_2) W^3_{LK}(z_3)\right) = \mathrm{Tr}_{\mathcal{H}}\left((-1)^{\mathrm{F}} e^{-\hbar P} W^1_{NI}(z_1) \widetilde{W}^2_{MJ}(z_2) W^3_{LK}(z_3)\right),}$$

as well as the Yangian.

- Considering the S-dual 6d theory, we may localize the path integral of its effective 5d theory to obtain an **'S-dual'** of Costello's 4d Chern-Simons theory where the coupling is inverted, i.e.,

$$\boxed{\int_\Gamma \mathcal{D}\mathcal{A}_{\widetilde{\alpha}0}\mathcal{D}\mathcal{A}_{z0}\mathcal{D}\mathcal{A}_{\bar{z}0} \prod_i \mathrm{Tr}(P\ e^{\int_{L_i}\mathcal{A}_0})\ e^{i\hbar\int_{\partial M} C\wedge\mathrm{Tr}\left(\mathcal{A}_0\wedge d\mathcal{A}_0+\frac{2}{3}\mathcal{A}_0\wedge\mathcal{A}_0\wedge\mathcal{A}_0\right)}}$$

which is a 4d analog of an 'S-dual' 3d analytically-continued Chern-Simons theory.

D4-brane worldvolume theory with NS5 boundary conditions

The low energy worldvolume theory of N coincident D4-branes on a flat manifold, M, involves fields which transform as reps. of $SO_M(5) \times SO_R(5)$:

$$\begin{aligned} A_M &: (\mathbf{5}, \mathbf{1}) \\ \phi_{\widehat{M}} &: (\mathbf{1}, \mathbf{5}) \\ \rho_{A\widehat{A}} &: (\mathbf{4}, \mathbf{4}) \end{aligned} \tag{7}$$

with the classical action

$$S = -\frac{1}{g_5^2}\int_M d^5x \, \mathrm{Tr}\,\Big(\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}D_M\phi_{\widehat{M}}D^M\phi^{\widehat{M}} + \frac{1}{4}[\phi_{\widehat{M}}, \phi_{\widehat{N}}][\phi^{\widehat{M}}, \phi^{\widehat{N}}] + i\rho^{A\widehat{A}}(\Gamma^M)_A{}^B D_M \rho_{B\widehat{A}} + \rho^{A\widehat{A}}(\Gamma^{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}}[\phi_{\widehat{M}}, \rho_{A\widehat{B}}]\Big),$$

i.e., 5d $\mathcal{N} = 2$ SYM.

It is invariant under the SUSY transformations

$$\begin{aligned}\delta A_M =&2\zeta^{A\widehat{A}}(\Gamma_M)_A{}^B\rho_{B\widehat{A}}\\ \delta\phi^{\widehat{M}} =&-i2\zeta^{A\widehat{A}}(\Gamma^{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}}\rho_{A\widehat{B}}\\ \delta\rho_{A\widehat{A}} =&(\Gamma^M)_A{}^B D_M\phi^{\widehat{M}}(\Gamma_{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}}\zeta_{B\widehat{B}} - \frac{i}{2}(\Gamma_{\widehat{M}})_{\widehat{A}}{}^{\widehat{B}}(\Gamma_{\widehat{N}})_{\widehat{B}\widehat{C}}[\phi^{\widehat{M}},\phi^{\widehat{N}}]\zeta_A{}^{\widehat{C}}\\ &-\frac{i}{2}F^{MN}(\Gamma_{MN})_{AB}\zeta^B_{\widehat{A}}.\end{aligned} \tag{8}$$

The stack of D4-branes shall be taken to end on an NS5-brane in the following type IIA brane configuration in flat Euclidean space

	1	2	3	4	5	6	7	8	9	10
D4	×	×	×	×	×					
NS5	×	×		×	×	×	×			

where, e.g., an empty entry under '3' indicates that the brane is located at $x^3 = 0$. The scalar fields $\{\phi_{\widehat{1}}, \phi_{\widehat{2}}, \phi_{\widehat{3}}, \phi_{\widehat{4}}, \phi_{\widehat{5}}\}$ are understood to parametrize the $\{6, 7, 8, 9, 10\}$ directions, respectively.

The configuration above implies the **boundary conditions**

$$F_{\mu 3} = 0|_{x^3=0}, \;\; D_3\phi^{\widehat{1},\widehat{2}} = 0|_{x^3=0}, \;\; \phi^{\widehat{3},\widehat{4},\widehat{5}} = 0|_{x^3=0}$$

(where μ is the boundary rotation group index), together with projection conditions on the fermionic fields.

Topological-holomorphic twist

4d Chern-Simons theory on $Y \times \Sigma$ is topological-holomorphic:

- It has **diffeomorphism invariance** along the 2-manifold denoted Y.
- It has **holomorphic dependence** on the Riemann surface, Σ.

To obtain it from the D4-NS5 system, we ought to perform a **partial twist** that has the above properties.

To this end, we note that $M = Y \times \mathbb{R}_+ \times \Sigma$, and we wish to twist the D4-brane worldvolume theory along $Y \times \mathbb{R}_+$.

This amounts to redefining the $SO_V(3)$ rotation group of $V = Y \times \mathbb{R}_+$ to be the diagonal subgroup

$$SO_V(3)' \subset SO_V(3) \times SO_R(3),$$

where $SO_R(3) \subset SO_R(5)$ rotates $\{\phi_{\widehat{1}}, \phi_{\widehat{2}}, \phi_{\widehat{3}}\}$.

Specifically, we are studying the following type IIA configuration:

	$\tilde{V}$									
	Y		$\mathbb{R}$	Σ		$N\tilde{V}\subset T^*\tilde{V}$				
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D4	$\times$	$\times$	$\times$	$\times$	$\times$					
NS5	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$			

The twist arises in this configuration because $V \subset \tilde{V} = Y \times \mathbb{R}$, where $\tilde{V}$ is the zero section of the cotangent bundle $T^*\tilde{V}$, and 'coordinates' normal to $\tilde{V}$ in $T^*\tilde{V}$ must be components of one-forms, as we shall obtain via twisting.[4]

4. M. Bershadsky, C. Vafa, V. Sadov, *D-branes and topological field theories*, *Nuclear Physics B* **463** (2-3) (1996) 420-434

Let us see now compute the partial twist. Having performed the reductions $SO_M(5) \to SO_V(3) \times SO_\Sigma(2)$ and $SO_R(5) \to SO_R(3) \times SO_R(2)$, we denote the relevant indices as

	$SO_V(3)$	$SO_R(3)$	$SO_\Sigma(2)$	$SO_R(2)$
Vector	$\alpha, \beta, \gamma, \ldots$	$\widehat{\alpha}, \widehat{\beta}, \widehat{\gamma}, \ldots$	$m, n, p, \ldots$	$\widehat{m}, \widehat{n}, \widehat{p}, \ldots$
Spinor	$\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \ldots$	$\widehat{\bar{\alpha}}, \widehat{\bar{\beta}}, \widehat{\bar{\gamma}}, \ldots$	$\bar{m}, \bar{n}, \bar{p}, \ldots$	$\widehat{\bar{m}}, \widehat{\bar{n}}, \widehat{\bar{p}}, \ldots$

Partial twisting amounts to setting the hatted $SO_R(3)$ indices to unhatted indices.

As a result, the scalar fields $\{\phi_{\widehat{1}}, \phi_{\widehat{2}}, \phi_{\widehat{3}}\}$ now transform as the components $\{\phi_1, \phi_2, \phi_3\}$ of a one-form on $Y \times \mathbb{R}_+$.

In addition, the spinor fields $\rho_{A\widehat{A}} = \rho_{\bar{\alpha}\bar{m}\widehat{\bar{\alpha}}\widehat{\bar{m}}}$ can be expanded after twisting as

$$\rho_{\bar{\alpha}\bar{m}\bar{\beta}\widehat{\bar{m}}} = \epsilon_{\bar{\alpha}\bar{\beta}}\eta_{\bar{m}\widehat{\bar{m}}} + (\sigma^\alpha)_{\bar{\alpha}\bar{\beta}}\psi_{\alpha\bar{m}\widehat{\bar{m}}}, \tag{9}$$

where $\eta_{\bar{m}\widehat{\bar{m}}}$ and $\psi_{\alpha\bar{m}\widehat{\bar{m}}}$ transform as **1** and **3** under $SO_V(3)'$.

Here we have used the antisymmetric matrix $\epsilon_{\bar{\alpha}\bar{\beta}}$ and the symmetric matrix $(\sigma^\alpha)_{\bar{\alpha}\bar{\gamma}} = (\sigma^\alpha)_{\bar{\alpha}}{}^{\bar{\beta}}\epsilon_{\bar{\beta}\bar{\gamma}}$, where ϵ is the Levi-Civita symbol and σ^α are the Pauli matrices.

Likewise, we can expand the SUSY transformation parameters $\zeta_{A\widehat{A}} = \zeta_{\bar{\alpha}\bar{m}\widehat{\bar{\alpha}}\widehat{\bar{m}}}$ as

$$\zeta_{\bar{\alpha}\bar{m}\bar{\beta}\widehat{\bar{m}}} = \epsilon_{\bar{\alpha}\bar{\beta}}\zeta_{\bar{m}\widehat{\bar{m}}} + (\sigma^\alpha)_{\bar{\alpha}\bar{\beta}}\zeta_{\alpha\bar{m}\widehat{\bar{m}}}. \tag{10}$$

Substituting these expansions into the SUSY transformations, we can obtain the partially twisted SUSY transformations.

However, we wish to pick a supercharge, Q, that is scalar along V, w.r.t. which the 5d theory is **topological-holomorphic**, i.e., correlation functions of Q-invariant local operators have holomorphic dependence on Σ and no other dependence on M.

There are in fact two such supercharges, that each satisfy both

$$\begin{aligned} \{Q, \ldots\} &\propto P_{\bar{z}}, \\ \{Q, \ldots\} &\propto P_{\beta}, \end{aligned} \tag{11}$$

and we shall pick one of them without loss of generality.[5]

5. In upcoming work which connects our present setup to the geometric Langlands program of Kapustin-Witten, we will select a linear combination of the two.

This supercharge generates the SUSY transformations

$$\begin{aligned}
Q\mathcal{A}_\alpha &= 0 & Q\eta_{11} &= 2iF_{z\bar{z}} + 2i[\phi_{\widehat{z}}, \phi_{\widehat{\bar{z}}}] - D_\beta \phi^\beta \\
Q\overline{\mathcal{A}}_\alpha &= -8\psi_{\alpha 22} & Q\eta_{12} &= 0 \\
QA_{\bar{z}} &= 0 & Q\eta_{21} &= 0 \\
QA_z &= 4\eta_{12} & Q\eta_{22} &= 4D_{\bar{z}}\phi_{\widehat{\bar{z}}} \\
Q\phi_{\widehat{\bar{z}}} &= 0 & Q\psi_{\alpha 11} &= -\frac{1}{2}\varepsilon^{\beta\gamma}{}_{\alpha}\mathcal{F}_{\beta\gamma} \\
Q\phi_{\widehat{z}} &= -4i\eta_{21} & Q\psi_{\alpha 12} &= 2\mathcal{D}_\alpha \phi_{\widehat{\bar{z}}} \\
& & Q\psi_{\alpha 21} &= -i2\mathcal{F}_{\alpha\bar{z}} \\
& & Q\psi_{\alpha 22} &= 0,
\end{aligned} \tag{12}$$

that satisfy $Q^2 = 0$ on-shell.

Here, we have defined the complex coordinates $z = x^4 + ix^5$ and $\bar{z} = x^4 - ix^5$, the complex gauge fields

$$\mathcal{A}_\alpha = A_\alpha + i\phi_\alpha, \quad \overline{\mathcal{A}}_\alpha = A_\alpha - i\phi_\alpha, \tag{13}$$

and

$$A_z = \frac{1}{2}(A_4 - iA_5), \quad A_{\bar{z}} = \frac{1}{2}(A_4 + iA_5), \tag{14}$$

whereby we have the covariant derivatives

$$\mathcal{D}_\alpha = \partial_\alpha + [\mathcal{A}_\alpha, \cdot\,], \quad \overline{\mathcal{D}}_\alpha = \partial_\alpha + [\overline{\mathcal{A}}_\alpha, \cdot\,], \tag{15}$$

and

$$D_z = \partial_z + [A_z, \cdot\,], \quad D_{\bar{z}} = \partial_{\bar{z}} + [A_{\bar{z}}, \cdot\,], \tag{16}$$

and the field strengths $\mathcal{F}_{\beta\gamma} = [\mathcal{D}_\beta, \mathcal{D}_\gamma]$, $\mathcal{F}_{\alpha z} = [\mathcal{D}_\alpha, D_z]$ and $F_{z\bar{z}} = [D_z, D_{\bar{z}}]$. We have also defined the scalar fields

$$\phi_{\widehat{z}} = \frac{1}{2}(\phi_{\widehat{4}} - i\phi_{\widehat{5}}), \quad \phi_{\overline{\widehat{z}}} = \frac{1}{2}(\phi_{\widehat{4}} + i\phi_{\widehat{5}}). \tag{17}$$

Note that $\mathcal{A}_\alpha$ and $A_{\bar{z}}$, are Q-**invariant**, which will prove essential to obtaining 4d CS theory.

Also, the Q-transformations are consistent with the partially twisted NS5 boundary conditions:

$$\begin{aligned} A_3 &= 0|_{x^3=0}, & \partial_3 A_\mu &= 0|_{x^3=0}, \\ \partial_3 \phi^{1,2} &= 0|_{x^3=0}, & \phi^{3,\widehat{4},\widehat{5}} &= 0|_{x^3=0}, \\ \partial_3 \eta_{1\widehat{m}} &= 0|_{x^3=0}, & \eta_{2\widehat{m}} &= 0|_{x^3=0}, \\ \psi_{\widetilde{\alpha}1\widehat{m}} &= 0|_{x^3=0}, & \partial_3 \psi_{\widetilde{\alpha}2\widehat{m}} &= 0|_{x^3=0}, \\ \partial_3 \psi_{31\widehat{m}} &= 0|_{x^3=0}, & \psi_{32\widehat{m}} &= 0|_{x^3=0}, \end{aligned} \tag{18}$$

where $\widetilde{\alpha} = 1, 2$.

The partially twisted action is

$$
\begin{aligned}
S_{twisted} = -\frac{1}{g_5^2}\int_M d^5x \, \mathrm{Tr}\Big(&\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{4}F_{\alpha n}F^{\alpha n} + \frac{1}{4}F_{m\beta}F^{m\beta} + \frac{1}{4}F_{mn}F^{mn} \\
&+ \frac{1}{2}D^{\alpha}\phi^{\beta}D_{\alpha}\phi_{\beta} + \frac{1}{2}D^{\alpha}\phi^{\widehat{n}}D_{\alpha}\phi_{\widehat{n}} \\
&+ \frac{1}{2}D^{m}\phi^{\beta}D_{m}\phi_{\beta} + \frac{1}{2}D^{m}\phi^{\widehat{n}}D_{m}\phi_{\widehat{n}} \\
&+ \frac{1}{4}[\phi^{\alpha},\phi^{\beta}][\phi_{\alpha},\phi_{\beta}] + \frac{1}{4}[\phi^{\alpha},\phi^{\widehat{n}}][\phi_{\alpha},\phi_{\widehat{n}}] \\
&+ \frac{1}{4}[\phi^{\widehat{m}},\phi^{\beta}][\phi_{\widehat{m}},\phi_{\beta}] + \frac{1}{4}[\phi^{\widehat{m}},\phi^{\widehat{n}}][\phi_{\widehat{m}},\phi_{\widehat{n}}] \\
&- 4\varepsilon^{\beta\gamma\alpha}\psi_{\alpha 11}\overline{\mathcal{D}}_{\beta}\psi_{\gamma 22} + 4\varepsilon^{\alpha\beta}{}_{\gamma}\psi^{\gamma}{}_{21}\mathcal{D}_{\alpha}\psi_{\beta 12} - i4\eta_{11}\mathcal{D}_{\alpha}\psi^{\alpha}{}_{22} \\
&- i4\psi^{\alpha}{}_{21}\overline{\mathcal{D}}_{\alpha}\eta_{12} + i4\psi^{\alpha}{}_{12}\overline{\mathcal{D}}_{\alpha}\eta_{21} + i4\eta_{22}\mathcal{D}_{\alpha}\psi^{\alpha}{}_{11} \\
&- i8\psi^{\gamma}{}_{21}D_{z}\psi_{\gamma 22} - i8\psi^{\gamma}{}_{11}D_{\bar{z}}\psi_{\gamma 12} + i8\eta_{22}D_{z}\eta_{21} + i8\eta_{11}D_{\bar{z}}\eta_{12} \\
&- 8\psi^{\beta}{}_{12}[\phi_{\widehat{z}},\psi_{\beta 22}] - 8\psi^{\beta}{}_{21}[\phi_{\widehat{\bar{z}}},\psi_{\beta 11}] + 8\eta_{22}[\phi_{\widehat{z}},\eta_{12}] + 8\eta_{11}[\phi_{\widehat{\bar{z}}},\eta_{21}]\Big).
\end{aligned}
$$

This action can be put in Q-**exact** $+$ Q-**invariant** form as follows. Firstly, the boson action is **equivalent** upon integration by parts to

$$-\frac{1}{g_5^2}\int_M d^5x\ \mathrm{Tr}\left(\frac{1}{4}\mathcal{F}^{\beta\gamma}\overline{\mathcal{F}}_{\beta\gamma} + 2\mathcal{F}^{\alpha}_{\bar{z}}\overline{\mathcal{F}}_{\alpha z} + 2\mathcal{D}^{\alpha}\phi_{\widehat{\bar{z}}}\overline{\mathcal{D}}_{\alpha}\phi_{\widehat{z}} + 8D_z\phi_{\widehat{z}}D_{\bar{z}}\phi_{\widehat{\bar{z}}} + \frac{1}{2}\left(2iF_{z\bar{z}} + 2i[\phi_{\widehat{z}},\phi_{\widehat{\bar{z}}}] - D_\beta\phi^\beta\right)^2\right) \tag{19}$$

To make $Q^2 = 0$ hold off-shell, we introduce an auxiliary field, d:

$$\begin{aligned} &-\frac{1}{g_5^2}\int_M d^5x\ \mathrm{Tr}\left(\frac{1}{2}\left(2iF_{z\bar{z}} + 2i[\phi_{\widehat{z}},\phi_{\widehat{\bar{z}}}] - D_\beta\phi^\beta\right)^2\right) \\ \to &-\frac{1}{g_5^2}\int_M d^5x\ \mathrm{Tr}\left(d\left(2iF_{z\bar{z}} + 2i[\phi_{\widehat{z}},\phi_{\widehat{\bar{z}}}] - D_\beta\phi^\beta\right) - \frac{1}{2}d^2\right), \end{aligned} \tag{20}$$

and modify the Q-transformations such that

$$Q\eta_{11} = d, \quad Qd = 0. \tag{21}$$

The action can then be put in Q-exact + Q-invariant form:

$$S_{twisted} = Q\Psi - \frac{1}{g_5^2}\int_M d^5x\ \mathrm{Tr}\Big(4\varepsilon^{\alpha\beta\gamma}\psi_{\gamma 21}\mathcal{D}_\alpha\psi_{\beta 12} + i2\varepsilon^{\alpha\rho\sigma}\eta_{22}\mathcal{D}_\alpha\chi_{\rho\sigma 11} - i4\varepsilon^{\gamma\rho\sigma}\chi_{\rho\sigma 11}D_{\bar{z}}\psi_{\gamma 12} - 4\varepsilon^{\beta\rho\sigma}\psi_{\beta 21}[\phi_{\widehat{\bar{z}}},\chi_{\rho\sigma 11}]\Big), \tag{22}$$

(where $\chi_{\rho\sigma 11} = \varepsilon_{\rho\sigma\alpha}\psi^{\alpha}{}_{11}$) with

$$\Psi = -\frac{1}{g_5^2}\int_M d^5x\ \mathrm{Tr}\Big(-\frac{1}{4}\chi^{\beta\gamma}{}_{11}\overline{\mathcal{F}}_{\beta\gamma} + i\psi^{\alpha}{}_{21}\overline{\mathcal{F}}_{\alpha z} + \psi^{\alpha}{}_{12}\overline{\mathcal{D}}_\alpha\phi_{\widehat{z}} + 2\eta_{22}D_z\phi_{\widehat{z}} + \eta_{11}\big(2iF_{z\bar{z}} + 2i[\phi_{\widehat{z}},\phi_{\widehat{\bar{z}}}] - D_\beta\phi^\beta\big) - \frac{1}{2}\eta_{11}d\Big). \tag{23}$$

Here, the dependence on the metric of $V = Y \times \mathbb{R}_+$ is **completely contained** in the Q-exact term, while there is **dependence on the complex structure** of Σ in the non-Q-exact part of the action.

Hence, along the boundary, $\partial M = Y \times \Sigma$, the partially twisted theory we have derived is topological along Y but depends on the complex structure of Σ, **just as in Costello's 4d Chern-Simons theory**.

Boundary action

The D4-brane worldvolume theory can be coupled to the RR 1-form, R, sourced by D0-branes, via

$$S_{top} = \frac{i}{g_5^2} \int_M R \wedge \mathrm{Tr}\,(F \wedge F). \tag{24}$$

We shall only be concerned with D0-branes that source R such that $dR = 0$. This allows us to write (24) as the **boundary action**

$$\frac{-i}{g_5^2} \int_{\partial M} R \wedge \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{25}$$

In order to include such a coupling without breaking the Q-supersymmetry, we shall first require that $R = C(z)dz$, i.e., it is **meromorphic** w.r.t. Σ. Then, the boundary action only depends on A_1, A_2 and $A_{\bar{z}}$, where only $A_{\bar{z}}$ is Q-invariant.

Hence, to obtain a Q-**invariant boundary action**, we must add boundary interaction terms involving ϕ_1 and ϕ_2 to the action such that its dependence on (A_1, A_2) is replaced by dependence on $(\mathcal{A}_1, \mathcal{A}_2)$, resulting in

$$S_{boundary} = \frac{-i}{g_5^2} \int_{\partial M} C \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big), \qquad (26)$$

where we have used the notation $A_{\bar{z}} = \mathcal{A}_{\bar{z}}$.

Localization to 4d Chern-Simons theory

To evaluate the path integral, we rescale the Q-exact part of the action by a large parameter, localizing the path integral to the **bosonic fixed points** of the Q-symmetry. We can then evaluate the path integral via **perturbation theory** around these fixed points.

Doing so, we arrive at

$$\boxed{\int_\Gamma \mathcal{D}\mathcal{A}_{\widetilde{\alpha}0}\mathcal{D}\mathcal{A}_{z0}\mathcal{D}\mathcal{A}_{\bar{z}0}\ e^{\frac{i}{g_5^2}\int_{\partial M} C\wedge \mathrm{Tr}\left(\mathcal{A}_0\wedge d\mathcal{A}_0+\frac{2}{3}\mathcal{A}_0\wedge\mathcal{A}_0\wedge\mathcal{A}_0\right)}} \tag{27}$$

This is the **partition function** of 4d CS theory with gauge group $U(N)_{\mathbb{C}} = GL(N, \mathbb{C})$ (where $\hbar = g_5^2$). Here, Γ is a **nonperturbative integration cycle** in field space, corresponding to the restriction to ∂M of the bosonic fixed points of the Q-symmetry, i.e.,

$$\boxed{\begin{aligned} \mathcal{F}_{\widetilde{\alpha}\widetilde{\beta}0} &= 0 \\ \mathcal{F}_{\widetilde{\alpha}\bar{z}0} &= 0 \\ 2i\mathcal{F}_{z\bar{z}0} - D_{\widetilde{\alpha}0}\phi_0^{\widetilde{\alpha}} &= 0 \end{aligned}} \tag{28}$$

(where $\widetilde{\alpha}, \widetilde{\beta} = 1, 2$). As shown recently,[6] one can also obtain an integration cycle for 4d CS theory by realizing it via a stack of D5-branes supported on the product of an Ω-background disk and $Y \times \Sigma$.

6. K. Costello, J. Yagi, *Unification of integrability in supersymmetric gauge theories*, arXiv:1810.01970

Wilson lines

Since the components of $\mathcal{A}$ along Y are Q-invariant, i.e., $Q\mathcal{A}_{\widetilde{\alpha}} = 0$, we can define **supersymmetric Wilson lines** along Y, i.e.,

$$W = \mathrm{Tr}(P\ e^{\int_{L\subset Y}\mathcal{A}}), \tag{29}$$

as observables of the 5d theory that we can define to be associated with representations of $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{C})$, or $\mathfrak{g}[[z]]$.

In string theory, such a Wilson line arises from the worldline of the boundary of a **fundamental string** ending on the D4-brane worldvolume.

Inserting such Wilson lines along $Y \subset \partial M$, the 5d theory localizes to

$$\int_\Gamma \mathcal{D}\mathcal{A}_{\widetilde{\alpha}0}\mathcal{D}\mathcal{A}_{z0}\mathcal{D}\mathcal{A}_{\bar{z}0}\prod_i \mathrm{Tr}\left(P\ e^{\int_{L_i}\mathcal{A}_0}\right)\ e^{\frac{i}{g_5^2}\int_{\partial M} C\wedge\mathrm{Tr}\left(\mathcal{A}_0\wedge d\mathcal{A}_0+\frac{2}{3}\mathcal{A}_0\wedge\mathcal{A}_0\wedge\mathcal{A}_0\right)}$$

For an **intersection** of perpendicular Wilson lines associated with representations of $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{C})$, this correlation function corresponds to an **R-matrix**. For a **lattice** of such Wilson lines, it is identified with the **partition function** of an integrable lattice model (as first argued by Costello).

Moreover, using three Wilson lines in reps. of $\mathfrak{g}$, **topological invariance** along Y implies the **Yang-Baxter equation**. For $\Sigma = \mathbb{C}$, replacing one of them with a Wilson line in a rep. of $\mathfrak{g}[[z]]$ gives us the **Yangian** in the form of the RTT relation.

T-duality as a lift to 6d

To categorify the R-matrix elements and the YBE with spectral parameter, we need to **lift** our 5d partially twisted theory to a 6d one.

From the string theory perspective this corresponds to **T-duality** in a direction transverse to the D4-branes. We shall pick this direction to be x^6, which leads to the following type IIB configuration

$$\overbrace{Y \times \mathbb{R}}^{V'} \quad \Sigma \quad S^1 \quad NV' \subset T^*V'$$

	Y		$\mathbb{R}$	Σ		S^1	$NV' \subset T^*V'$			
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D5	×	×	×	×	×	×				
NS5	×	×		×	×	×	×			

The lift along the x^6 direction implies that ϕ_1 is replaced by the gauge field component A_6, and thus

$$\mathcal{A}_1 = A_1 + i\phi_1 \to A_1 + iA_6 = 2A_{\overline{w}} \tag{30}$$
$$\overline{\mathcal{A}}_1 = A_1 - i\phi_1 \to A_1 - iA_6 = 2A_w, \tag{31}$$

where we have defined the complex coordinates $w = x^1 + ix^6$ and $\overline{w} = x^1 - ix^6$. We first use (30) and $\partial_1 \to 2\partial_{\overline{w}}$,[7] to lift the boundary action:

$$\frac{-i}{g_5^2}\int_{\partial M} C \wedge \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)$$
$$\to \frac{-1}{g_5^2(2\pi r)}\int_{\partial M\times S^1} dw \wedge C \wedge \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right),$$

(using $A_{\overline{w}} = \mathcal{A}_{\overline{w}}$) where r is the radius of the x^6 circle.

7. This choice is necessary for gauge invariance.

To lift the 5d Q-transformations and bulk action, we use

$$\begin{aligned} \mathcal{D}_1 &\to 2D_{\overline{w}} \\ \overline{\mathcal{D}}_1 &\to 2D_w. \end{aligned} \tag{32}$$

The Q-transformations we obtain this way are

$$\begin{aligned} Q\mathcal{A}_i &= 0 \\ Q\overline{\mathcal{A}}_i &= -8\psi_{i22} \\ QA_{\overline{w}} &= 0 \\ QA_w &= -4\widetilde{\eta}_{22} \\ QA_{\bar{z}} &= 0 \\ QA_z &= 4\eta_{12} \\ Q\phi_{\widehat{\bar{z}}} &= 0 \\ Q\phi_{\widehat{z}} &= -4i\eta_{21} \\ Qd &= 0 \end{aligned} \qquad \begin{aligned} Q\eta_{11} &= d \\ Q\eta_{12} &= 0 \\ Q\eta_{21} &= 0 \\ Q\eta_{22} &= 4D_{\bar{z}}\phi_{\widehat{\bar{z}}} \\ Q\widetilde{\psi}_{j11} &= 2\mathcal{F}_{j\overline{w}} \\ Q\chi_{ij11} &= -\mathcal{F}_{ij} \end{aligned} \qquad \begin{aligned} Q\widetilde{\eta}_{12} &= 4D_{\overline{w}}\phi_{\widehat{\bar{z}}} \\ Q\psi_{i12} &= 2\mathcal{D}_i\phi_{\widehat{\bar{z}}} \\ Q\widetilde{\eta}_{21} &= -i4F_{\overline{w}\bar{z}} \\ Q\psi_{i21} &= -i2\mathcal{F}_{i\bar{z}} \\ Q\widetilde{\eta}_{22} &= 0 \\ Q\psi_{i22} &= 0 \end{aligned}$$

$i, j, k, \ldots = 2, 3,\ \chi_{1j11} = \widetilde{\psi}_{j11},\ \psi_{112} = \widetilde{\eta}_{12},\ \psi_{121} = \widetilde{\eta}_{21},\ \psi_{122} = \widetilde{\eta}_{22}.$

The bulk action we obtain is

$$S^{6d}_{twisted} = Q\Psi' - \frac{1}{g_5^2 2\pi r}\int_{M\times S^1} d^6x \,\mathrm{Tr}\,\Big(8\varepsilon^{jk}\psi_{k21}D_{\overline{w}}\psi_{j12} - 4\varepsilon^{ik}\psi_{k21}\mathcal{D}_i\widetilde{\eta}_{12} + 4\varepsilon^{ij}\widetilde{\eta}_{21}\mathcal{D}_i\psi_{j12}$$
$$+ i4\varepsilon^{jk}\eta_{22}D_{\overline{w}}\chi_{jk11} - i4\varepsilon^{ij}\eta_{22}\mathcal{D}_i\widetilde{\psi}_{j11}$$
$$- i4\varepsilon^{jk}\chi_{jk11}D_{\overline{z}}\widetilde{\eta}_{12} + i8\varepsilon^{ij}\widetilde{\psi}_{j11}D_{\overline{z}}\psi_{i12}$$
$$- 4\varepsilon^{jk}\widetilde{\eta}_{21}[\phi_{\overline{\widehat{z}}}, \chi_{jk11}] + 8\varepsilon^{ij}\psi_{i21}[\phi_{\overline{\widehat{z}}}, \widetilde{\psi}_{j11}]\Big),$$

where

$$\Psi' = -\frac{1}{g_5^2 2\pi r}\int_{M\times S^1} d^6x \,\mathrm{Tr}\,\Big(-\frac{1}{4}\chi^{ij}{}_{11}\overline{\mathcal{F}}_{ij} + \widetilde{\psi}^j_{11}\overline{\mathcal{F}}_{jw} + i\psi^i{}_{21}\overline{\mathcal{F}}_{iz} + i2\widetilde{\eta}_{21}F_{wz}$$
$$+ \psi^i{}_{12}\overline{\mathcal{D}}_i\phi_{\widehat{z}} + 2\widetilde{\eta}_{12}D_w\phi_{\widehat{z}} + 2\eta_{22}D_z\phi_{\widehat{z}}$$
$$+ \eta_{11}(2iF_{z\overline{z}} + 2iF_{w\overline{w}} + 2i[\phi_{\widehat{z}}, \phi_{\overline{\widehat{z}}}] - D_j\phi^j) - \frac{1}{2}\eta_{11}d\Big).$$

This is a **topological-holomorphic** theory - the dependence of the metric in the x^2 and x^3 directions is **completely contained** in the Q-exact term, while the remaining terms **depend on the complex structures of Σ and Σ'**, where Σ' is the complex surface parametrized by w and $\overline{w}$.

Moreover, the boson action is equivalent to

$$
\begin{aligned}
S^{6d}_{boson} = -\frac{1}{g_5^2 2\pi r}\int_{M\times S^1} d^6x \ \mathrm{Tr}\bigg(&\frac{1}{4}F_{xy}F^{xy} + \frac{1}{4}F_{iy}F^{iy} + \frac{1}{4}F_{xj}F^{xj} + \frac{1}{4}F_{ij}F^{ij}\\
&+ \frac{1}{2}D^i\phi^j D_i\phi_j + \frac{1}{2}D^i\phi^{\widehat{n}}D_i\phi_{\widehat{n}}\\
&+ \frac{1}{2}D^x\phi^j D_x\phi_j + \frac{1}{2}D^x\phi^{\widehat{n}}D_x\phi_{\widehat{n}}\\
&+ \frac{1}{4}[\phi^i,\phi^j][\phi_i,\phi_j] + \frac{1}{4}[\phi^i,\phi^{\widehat{n}}][\phi_i,\phi_{\widehat{n}}]\\
&+ \frac{1}{4}[\phi^{\widehat{m}},\phi^j][\phi_{\widehat{m}},\phi_j] + \frac{1}{4}[\phi^{\widehat{m}},\phi^{\widehat{n}}][\phi_{\widehat{m}},\phi_{\widehat{n}}]\bigg),
\end{aligned}
$$

where $x, y = z, \bar{z}, w, \overline{w}$. Hence, the 6d theory we obtain by lifting is 6d $\mathcal{N} = (1,1)$ SYM **partially twisted** along the x^2 and x^3 directions, with a boundary coupling to the RR 2-form $idw \wedge C$.

One ought to be able to localize this partially twisted theory to a 5d Chern-Simons theory on the boundary with an action

$$S^{6d}_{boundary} = \frac{-1}{g_5^2(2\pi r)} \int_{\partial M \times S^1} dw \wedge C \wedge \mathrm{Tr}\Big(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\Big) \tag{33}$$

with nonperturbative integration cycle Γ' defined by

$$\begin{aligned} \mathcal{F}_{2\bar{z}0} &= 0 \\ \mathcal{F}_{2\overline{w}0} &= 0 \\ \mathcal{F}_{\overline{w}\bar{z}0} &= 0 \\ 2i\mathcal{F}_{z\bar{z}0} + 2i\mathcal{F}_{w\overline{w}0} - D_{20}\phi_0^2 &= 0. \end{aligned}$$

For $C = dz$, (33) is the commutative limit of Costello's 5d non-commutative Chern-Simons theory (where B-field was on).[8]

8. K. Costello, *M-theory in the Omega-background and 5-dimensional non-commutative gauge theory*, arXiv:1610.04144

S-duality

Next, we apply **S-duality**, whereby we obtain

		V'								
	Y		$\mathbb{R}$	Σ		S^1	$NV' \subset T^*V'$			
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
NS5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$				
D5	$\times$	$\times$		$\times$	$\times$	$\times$	$\times$			

$$\text{F-strings (Wilson lines)} \rightarrow \text{D1-branes (Wilson lines)}$$

$$\text{D5 bulk coupling- } (2\pi)^3 g\alpha' \rightarrow \text{NS5 bulk coupling- } (2\pi)^3\alpha'$$

$$\text{D5 bdry. coupling- } g_5^2(2\pi r) \rightarrow \text{NS5 bdry. coupling- } (2\pi)^5(\alpha')^2 r^{-1}/g_5^2$$

$$\text{RR 2-form- } i dw \wedge C \rightarrow \text{NS-NS 2-form- } i dw \wedge C$$

We shall use the S-dual partially twisted 6d $\mathcal{N} = (1,1)$ worldvolume theory of NS5-branes to **categorify** the R-matrix elements.

In other words, we want to describe the R-matrix elements in terms of the **Hilbert space** of a 6d theory.

The S-dual theory is used because it allows us to write its path integral as a trace of an expression with an obvious **expansion in positive powers of** $\hbar$, which the R-matrix is known to admit in the semi-classical limit.

Hilbert space of 6d theory and categorification

Taking the sixth dimension S^1 to be the Euclidean time dimension, we find that the Hilbert space of states, $\mathcal{H}$, of the Q-cohomology of the S-dual 6d theory, is given by the **Floer cohomology** corresponding to the BPS equations

$$\boxed{\begin{aligned} \mathcal{F}_{ij} &= 0 \\ \mathcal{F}_{i\bar{z}} &= 0 \\ \mathcal{F}_{i\overline{w}} &= 0 \\ F_{\overline{w}\bar{z}} &= 0 \\ 2iF_{z\bar{z}} + 2iF_{w\overline{w}} - D_j\phi^j &= 0 \end{aligned}} \tag{34}$$

which are the BPS equations of the partially twisted D5-brane worldvolume theory.

These are tunnelling equations that interpolate perturbative states given by solutions of the 5d equations

$$\boxed{\begin{aligned} \mathcal{F}_{\beta\gamma} &= 0 \\ \mathcal{F}_{\alpha\bar{z}} &= 0 \\ 2iF_{z\bar{z}} - D_\beta\phi^\beta &= 0 \end{aligned}} \tag{35}$$

which are the BPS equations from our study of the D4-brane worldvolume theory.

We can write the path integral of the S-dual 6d theory as a trace over the Hilbert space, $\mathcal{H}$. If we consider two perpendicular Wilson lines along Y in the D4-NS5 system which give rise to the R-matrix, T- and S-duality give

$$\boxed{R^{12}_{IK,JL}(z_1,z_2) = \mathrm{Tr}_{\mathcal{H}}\left((-1)^{\mathrm{F}} e^{-\hbar P} W^1_{IJ}(z_1) W^2_{KL}(z_2)\right)} \tag{36}$$

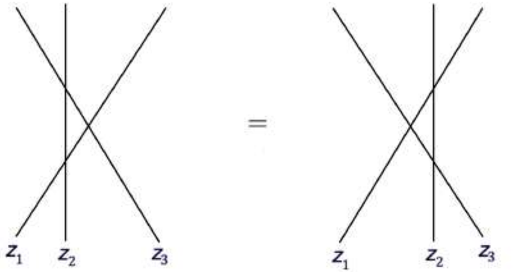
where z_1 and z_2 are the positions of the Wilson lines (denoted as W^1 and W^2) on Σ, I, J, K, L are basis elements of the representations of the Wilson lines, F is the fermion number operator, and where the operator

$$\boxed{P = i\int_{\partial M} C \wedge \mathrm{Tr}\left(\mathcal{A}\wedge d\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\right)} \tag{37}$$

In other words, *we have identified a vector space with each R-matrix element, whereby the vector space is the Floer cohomology of a set of 6d partial differential equations*.

In principle, we may compute each R-matrix element from the solutions of these equations.

- The Yang-Baxter equation can be realized in the 4d CS theory on the boundary due to the topological symmetry along Y.



- No singular behaviour arises in moving a Wilson line, as long as z_1, z_2 and z_3 are distinct.

We may also **categorify the Yang-Baxter equation** in this manner. Realizing the Yang-Baxter equation

$$\sum_{O,P,Q} R^{12}_{NM,QO}(z_1,z_2)R^{13}_{QL,IP}(z_1,z_3)R^{23}_{OP,JK}(z_2,z_3) = \sum_{R,S,T} R^{23}_{ML,RT}(z_2,z_3)R^{13}_{NT,SK}(z_1,z_3)R^{12}_{SR,IJ}(z_1,z_2)$$

in the original D4-NS5 system using Wilson lines labelled by spectral parameters z_1, z_2 and z_3, T- and S-duality on both sides of the equation give

$$\boxed{\mathrm{Tr}_{\mathcal{H}}((-1)^{\mathrm{F}} e^{-\hbar P} W^1_{NI}(z_1) W^2_{MJ}(z_2) W^3_{LK}(z_3)) = \mathrm{Tr}_{\mathcal{H}}((-1)^{\mathrm{F}} e^{-\hbar P} W^1_{NI}(z_1) \widetilde{W}^2_{MJ}(z_2) W^3_{LK}(z_3))}$$

We can also **categorify the Yangian** (in the form of the RTT relation) associated with rational integrable lattice models in an analogous manner, which also leads to an expression of the form above.

S-dual 4d Chern-Simons theory

When applying T-duality to the D4-NS5 system, we applied it along a very large circle in the x^6 direction, which resulted in a very small circle in the D5-brane worldvolume.

Hence, at low energies which define the Q-cohomology, the 6d worldvolume theory (and its S-dual) can be effectively regarded as the 5d theory obtained via dimensional reduction.

Considering the S-dual 6d theory, we may localize the path integral of its effective 5d theory to obtain

$$\boxed{\int_{\Gamma} \mathcal{D}\mathcal{A}_{\widetilde{\alpha}0}\mathcal{D}\mathcal{A}_{z0}\mathcal{D}\mathcal{A}_{\bar{z}0} \prod_i \mathrm{Tr}(P\ e^{\int_{L_i}\mathcal{A}_0})\ e^{i\hbar\int_{\partial M} C\wedge \mathrm{Tr}\left(\mathcal{A}_0\wedge d\mathcal{A}_0+\frac{2}{3}\mathcal{A}_0\wedge\mathcal{A}_0\wedge\mathcal{A}_0\right)}}$$

Hence, we have an **'S-dual'** of Costello's 4d Chern-Simons theory, where

$$\boxed{\frac{1}{\hbar} \to \hbar.} \tag{38}$$

This can be compared with the S-duality of 3d analytically-continued Chern-Simons theory,[9] which arises as a consequence of the S-duality of 4d $\mathcal{N} = 4$ SYM.

9. Y. Terashima, M. Yamazaki, *$SL(2,\mathbb{R})$ Chern-Simons, Liouville, and Gauge Theory on Duality Walls*, *JHEP* (2011) (8), 135
T. Dimofte, S. Gukov, *Chern-Simons theory and S-duality*, *JHEP* (2013)(5), 109

Conclusion and Future Directions

- We have made use of string theory to derive an integration cycle that allows us to define Costello's 4d CS theory nonperturbatively.

- We have also made use of string theory to categorify various objects associated with the lattice models the 4d CS theory physically realizes, in terms of Floer cohomology defined by some 6d differential equations. This is both physically and mathematically novel.

- As an offshoot, we were also able to derive an 'S-dual' 4d CS theory, thereby generalizing S-duality in 3d analytically-continued CS theory.

- Future work involves including D2-branes to realize surface defects in the 4d CS theory, which then allows us to do the same for integrable field theories, where the Lax operator can be derived from the Hilbert space of a 6d theory etc..

Thank you for your attention!