Isomonodromic deformations on torus, toric conformal blocks and gauge theories

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Based on works with Giulio Bonelli, Fabrizio Del Monte and Alessandro Tanzini: arXiv:1901.10497 [hep-th], arXiv:190*.***** [hep-th]

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Linear systems and their monodromies

- Fuchsian system on surface Σ:

$$d_z Y(z) = A(z)Y(z)$$

$d_z = dz\frac{\partial}{\partial z}$ — de Rham differential, $A(z) \in Mat_{N\times N}$ — meromorphic connection 1-form in some (possibly non-trivial) bundle $E \to \Sigma$, $Y(z) \in Mat_{N\times N}$ — multi-valued matrix of flat sections of E.

- It might be useful to choose such trivialization of E that $A(z)$ becomes non-single-valued (let's call this "twists"):

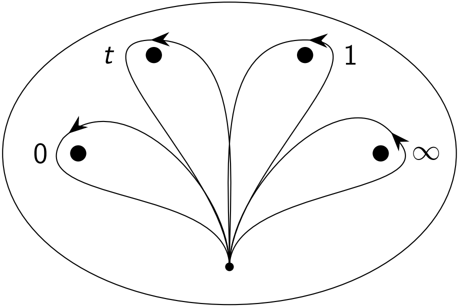
$$A(\gamma.z) = T_\gamma^{-1} A(z) T_\gamma$$

- Monodromies:

$$Y(\gamma.z) = T_\gamma Y(z) M_\gamma$$

Examples: sphere with 4 punctures

$$\frac{dY(z)}{dz} = \sum_{k=1}^{n} \frac{A_k}{z - z_k} Y(z)$$

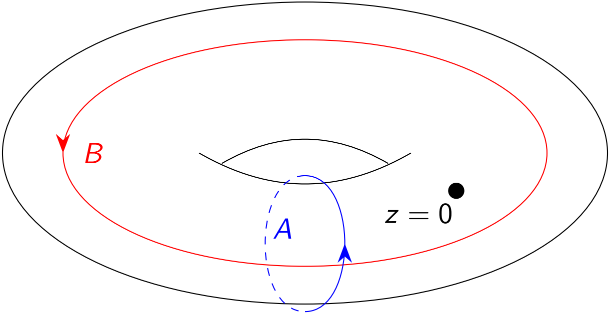


- Monodromy data:

$$\{M_0, M_1, M_t, M_\infty \in SL(N) | M_{z_1} M_{z_2} M_{z_3} M_\infty = 1\}/SL_N$$

- Complex (isomonodromic) modulus: z.
  For n punctures there are $n - 3$ moduli.

Example: torus with one puncture, $N = 2$



$$\partial_z Y(z|\tau) = L(z|\tau) Y(z|\tau), \quad L(z|\tau) = \begin{pmatrix} p & m x(2Q, z) \\ m x(-2Q, z) & -p \end{pmatrix}$$

$$T_A = \mathbb{I}_2, \qquad T_B = e^{2\pi i \boldsymbol{Q}}, \qquad x(u, z) = \frac{\theta_1(z-u|\tau)\theta_1'(\tau)}{\theta_1(z|\tau)\theta_1(u|\tau)}$$

Residue of $L(z|\tau)$ is fixed by m. p, Q are free parameters.

Example: torus with one puncture

- Monodromies of the solution ($M_0 = M_A M_B M_A^{-1} M_B^{-1}$):

$$Y(\gamma_A \cdot z|\tau) = Y(z) M_A, \quad Y(\gamma_B \cdot z|\tau) = e^{2\pi i \boldsymbol{Q}} Y(z) M_B$$

- Complex modulus: τ. For torus with n punctures we have extra $n-1$ moduli $z_2, ..., z_n$.
- Moduli of the SL_N fiber bundle: $\boldsymbol{Q} = \mathrm{diag}(Q_1, \ldots, Q_N)$ with constraint $\sum Q_i = 0$.

Monodromy map (for several punctures)

$$(\tau, \{z_\nu\}|\boldsymbol{Q}, d_z - A(z)) \longleftrightarrow (\tau, \{z_\nu\}|\{M_k\})$$

For single puncture:

$$(\tau|\boldsymbol{Q}, d_z - A(z)) \longleftrightarrow (\tau|M_A, M_B)$$

Isomonodromic deformations

- We keep monodromies fixed $\{M_k\}$ and consider them as initial data of the problem.
- We wish to study the dependence of moduli of the bundle $\boldsymbol{Q}$ and of connection $d_z - A(z)$ on the complex moduli.
- This gives rise to non-linear equations.

Example. Torus with 1 puncture and 2×2 system:

$$\begin{cases} \partial_z Y = LY, \\ 2\pi i \partial_\tau Y = -MY \end{cases} \qquad M = m \begin{pmatrix} \wp(2Q) & \partial_Q x(2Q, z) \\ \partial_Q x(-2Q, z) & \wp(2Q) \end{pmatrix}$$

Compatibility condition $2\pi i \partial_\tau L + \partial_z M + [M, L] = 0$ gives rise to

$$(2\pi i)^2 \frac{d^2 Q}{d\tau^2} = m^2 \wp'(2Q|\tau), \quad p = 2\pi i \partial_\tau Q$$

— non-autonomous 2-particle Calogero model.

Hamiltonians and tau-functions

- In the general situation isomonodromic equations are non-autonomous compatible equations in complex moduli.
- They are Hamiltonian with Hamiltonians given by

$$H_\tau = \frac{1}{2}\oint_A \frac{A(z)^2}{dz}, \quad H_{z_k} = \frac{1}{2}\oint_{z_k} \frac{A(z)^2}{dz}$$

- There are two separate conditions: $\partial_x H_y = \partial_y H_x$, $\{H_x, H_y\} = 0 \Rightarrow$

There exists tau function defined by

$$\partial_x \log \mathcal{T}(\tau, \{z_k\}) = H_x$$

For torus with 1 puncture and 2×2 system:

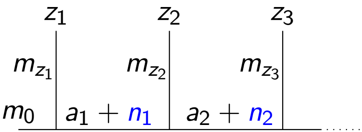
$$\partial_\tau \log \mathcal{T}(\tau) = (2\pi i \partial_\tau Q)^2 - m^2 \wp(2Q|\tau) - 2m^2\eta_1(\tau)$$

Isomonodromy-CFT-gauge theory correspondence (sphere)

Gamayun-Iorgov-Lisovyy (for central charge $c = 1$):

- Singular points $\leftrightarrow$ insertions of Virasoro vertex operators.
- Monodromies $\leftrightarrow$ exponents of Virasoro charges: $\Delta_\nu = m_\nu^2$, $M_\nu \sim \mathrm{diag}\left(e^{2\pi i m_\nu}, e^{-2\pi i m_\nu}\right)$. There is a shift $m_\nu \mapsto m_\nu + 1$.
- Tau function $\leftrightarrow$ Fourier sum of conformal blocks (Kyiv formula):

$$\mathcal{T}(\{m_\nu\}, \{a_k\}, \{\eta_k\}, \{z_k\}) = \sum_{\{n_i\} \in \mathbb{Z}^{n-3}} e^{2\pi i (\vec{n}, \vec{\eta})}$$



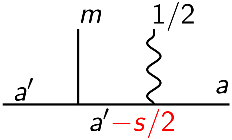
Alday-Gaiotto-Tachikawa & Nekrasov-Okounkov:

- Virasoro conformal block with appropriate normalization $\leftrightarrow$ partition function of $\mathcal{N} = 2$ $SU(2)$ quiver gauge theory.
- $\mathcal{T}(\ldots) =$ (Generalization of the) dual Nekrasov partition function.

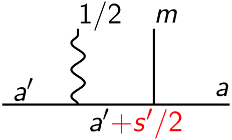
General strategy of the proof on sphere [GIL, IL-Teschner]

1. Consider conformal block with two degenerate $\phi_{(1,2)}$'s in x and x_0:

2. Compute analytic continuation in x



$$= \sum_{s'} \tilde{F}_{s's}(a', m, a)$$



3. Diagonalize shift operators $\nabla_k : a_k \mapsto a_k + 1$ by discrete Fourier transform. It turns ∇_k into numbers $e^{2\pi i \eta_k}$.

4. Prove that

$$\frac{\langle \phi_i(x)\bar{\phi}_j(x_0)V_1(z_1)\ldots V_n(z_n)\rangle}{\langle V_1(z_1)\ldots V_n(z_n)\rangle} = (x - x_0)^{-1/4}[Y(x_0)Y(x)]_{ji}$$

General strategy of the proof on sphere [GIL, IL-Teschner]

5. Construct energy-momentum tensor from degenerate field and get

$$\frac{\langle T(x)V_1(z_1)\dots V_n(z_n)\rangle}{\langle V_1(z_1)\dots V_n(z_n)\rangle} = \frac{1}{2}\frac{\mathrm{tr}\, A(x)^2}{dx^2}$$

6. Apply the OPE of $T(x)$ with $V(z)$

$$T(x)V_k(z_k) = \frac{\frac{1}{2}\,\mathrm{tr}\, A_k^2}{(z-z_k)^2}V_k(z_k) + \frac{1}{z-z_k}\partial V_k(z_k) + reg.$$

and prove the formula for the tau function

$$\mathcal{T} = const \times \langle V_1(z_1)\dots V_n(z_n)\rangle$$

Peculiarities in the toric case

1. One should add one $U(1)$ boson and switch from $\phi_{(1,2)}$ to fermions.
2. Computation of B-cycle monodromy switches the parity of Virasoro and $U(1)$ charges simultaneously.
3. Fourier transform also involves $U(1)$ boson, since there is a closed loop with unconstrained $U(1)$ charge.
4. Diagonalization of shifts will not work without extra boson.
5. 2-fermionic correlator is different:

$$\frac{1}{\mathcal{T}}\langle V_m(0)\bar{\psi}(z_0)\otimes\psi(z)\rangle = Y^{-1}(z_0)\mathrm{diag}\left(x(\rho-Q,z-z_0),x(\rho+Q,z-z_0)\right)Y(z)$$

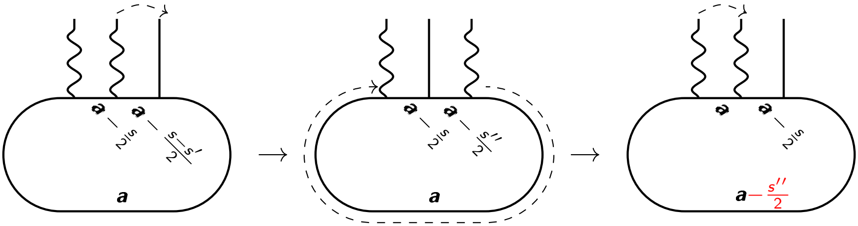
6. Relation for the tau function is different as well:

$$Z^D(\tau)=\eta(\tau)^{-2}\theta_1(\rho+Q(\tau))\theta_1(\rho-Q(\tau))\mathcal{T}(\tau)$$

7. * Bonus: solution of the non-autonomous equation can be written explicitly in terms of dual Nekrasov partition functions.

Fermions vs degenerate fields, or GL_N vs SL_N

- The main problem:

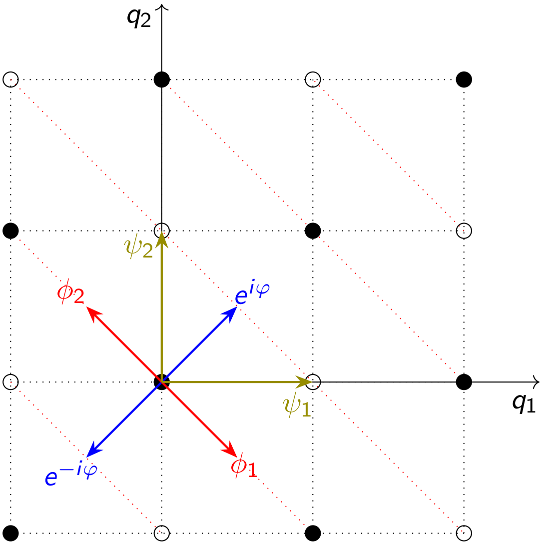


B-cycle monodromy shifts $a \mapsto a \pm \frac{1}{2}$, this breaks A-cycle monodromy: $M_A \mapsto -M_A$. Half-integer Fourier transformation does not help.

- Single free boson has the same problem.
- Fermionization formulas [ILT]:

$$\psi_i(z) = e^{i\varphi(z)}\phi_i(z), \quad \bar{\psi}_i(z) = e^{-i\varphi(z)}\bar{\phi}_i(z)$$

Bosonic/fermionic sectors



- Virasoro and $U(1)$ charges: $q^{Vir} = \frac{1}{2}(q_1 - q_2)$, $q^{U(1)} = q_1 + q_2$.
- Sectors differ by $q^{U(1)} \mod 2$ — extra shift of monodromy.
- In the $N \times N$ case in the same way we have N sectors distinguished by $q^{U(1)} \mod N$.

Fourier transformation and monodromies

- To be able to pick a concrete sector we add Fourier parameter ρ for $U(1)$ boson:

$$\langle V_m(0)\bar{\psi}(z_0)\otimes\psi(z)\rangle = \sum_{n,k} e^{\frac{in\eta}{2}} e^{4\pi i(\rho+\frac{1}{2})(\frac{n}{2}+k)}\Psi(z,z_0;m,a+\frac{n}{2},\frac{n+1}{2}+k)$$

  Here Ψ is $U(1)\times Vir$ toric conformal block with two fermion insertions, $a+\frac{n}{2}$ is Virasoro charge, $\frac{n+1}{2}+k$ is $U(1)$ charge.

- Compute monodromies:

$$\hat{M}_A = e^{-2\pi i \boldsymbol{a}} \equiv M_A,$$
$$\hat{M}_B = e^{i\frac{\eta}{2}\sigma^z+2\pi i\rho} e^{i\pi\boldsymbol{a}} F(a,m-1/2,a) e^{-i\pi\boldsymbol{a}} \equiv e^{2\pi i\rho} M_B.$$

  Here the Virasoro braiding matrix F is

$$F(a',m,a) = \begin{pmatrix} \frac{\cos\pi(m+a+a')}{\sin 2\pi a} & \frac{\cos\pi(m+a'-a)}{\sin 2\pi a} \\ -\frac{\cos\pi(m+a-a')}{\sin 2\pi a} & -\frac{\cos\pi(m-a-a')}{\sin 2\pi a} \end{pmatrix}.$$

Fermionic kernel

Denote

$$\frac{\langle V_m(0)\bar{\psi}(z_0)\otimes\psi(z)\rangle}{\langle V_m(0)\rangle} \equiv K(z_0,z)$$

Find such 2×2 kernel $K(z_0,z)$ that:

1. When $z\to z_0$: $K(z_0,z)\sim\frac{\mathbb{I}}{z_0-z}+reg.$
2. $K(z_0,z+1)=K(z_0,z)M_A,\quad K(z_0,z+\tau)=K(z_0,z)M_B e^{2\pi i\rho}$
3. $K(z_0+1,z)=M_A^{-1}K(z_0,z),\quad K(z_0+\tau,z)=e^{-2\pi i\rho}M_B^{-1}K(z_0,z)$

Such kernel exists and unique:

$$K(z_0,z)=Y^{-1}(z_0)\mathrm{diag}\left(x(\rho-Q,z-z_0),x(\rho+Q,z-z_0)\right)Y(z)$$

Unlike $Y(z)$, it represents a meromorphic section of a *trivial* bundle on $\Sigma\times\Sigma$.

From fermionic correlator to tau function

- Fermionic energy-momentum tensor:

$$T(z) = \frac{1}{2} \sum_{\gamma} \left(: \partial\bar{\psi}_{\gamma}(z)\psi_{\gamma}(z) : + : \partial\psi_{\gamma}(z)\bar{\psi}_{\gamma}(z) : \right)$$

- We take it from the OPE when $z \to z_0$, $z + z_0 = 2z_c$:

$$\frac{\langle T(z_c)V_m(0)\rangle}{\langle V_m(0)\rangle} = \frac{1}{2}\operatorname{tr}\left(L^2(z_c) + \frac{\theta_1''(\widetilde{\boldsymbol{Q}}|\tau)}{\theta_1(\widetilde{\boldsymbol{Q}}|\tau)} + 2L(z_c)\frac{\theta_1'(\widetilde{\boldsymbol{Q}}|\tau)}{\theta(\widetilde{\boldsymbol{Q}}|\tau)} - \frac{1}{3}\frac{\theta_1'''(\tau)}{\theta_1'(\tau)} \right)$$

- After some transformations we get:

$$\frac{\langle V_m(0)T(z_c)\rangle}{\langle V_m(0)\rangle} = \frac{1}{2}\operatorname{tr} L^2(z_c) + 2\pi i\partial_\tau \log\left(\eta(\tau)^{-2}\theta_1(\rho + Q(\tau))\theta_1(\rho - Q(\tau))\right)$$

From fermionic correlator to tau function

Taking the Ward identity

$$2\pi i\partial_\tau \log\langle V_m(0)\rangle = \oint_A dz \frac{\langle V_m(0)T(z)\rangle}{\langle V_m(0)\rangle}$$

we have

$$\partial_\tau \log\langle V_m(0)\rangle = \partial_\tau \log \mathcal{T} + \partial_\tau \left(\eta(\tau)^{-2}\theta_1(\rho + Q(\tau))\theta_1(\rho - Q(\tau))\right)$$

and finally

The main isomonodromy-CFT-gauge theory relation in the toric case:

$$Z^D(\tau) = \langle V_m(0)\rangle = const \times \eta(\tau)^{-2}\theta_1(\rho + Q(\tau))\theta_1(\rho - Q(\tau))\mathcal{T}(\tau)$$

Decoupling $U(1)$ boson back

- Introduce two sums of *Virasoro* conformal blocks only over *Virasoro* charges:

$$Z^D_{\epsilon/2}(\eta, a, m, \tau) = \sum_{n \in \mathbb{Z} + \frac{\epsilon}{2}} e^{in\eta} \operatorname{tr}_{\mathcal{V}_{a+n}} (q^{L_0} V_m(0)).$$

- Separate terms corresponding to even and odd lattices:

$$Z^D(\tau, \rho) = -Z^D_0(\tau)\eta(\tau)^{-1}\theta_2(2\rho|2\tau) + Z^D_{1/2}(\tau)\eta(\tau)^{-1}\theta_3(2\rho|2\tau)$$

- Do the same thing with the r.h.s:

$$Z^D(\tau, \rho) = \mathcal{T}(\tau)\eta(\tau)^{-2}\left(-\theta_2(2\rho|2\tau)\theta_3(2Q|2\tau) + \theta_3(2\rho|2\tau)\theta_2(2Q|2\tau)\right)$$

Equate the two things and get formulas for Virasoro Fourier sums:

$$Z^D_0(\tau) = \eta(\tau)^{-1}\theta_3(2Q|2\tau)\mathcal{T}(\tau)\,, \quad Z^D_{1/2}(\tau) = \eta(\tau)^{-1}\theta_2(2Q|2\tau)\mathcal{T}(\tau)$$

Bonus: solution of the non-autonomous Calogero model

- Just divide last two formulas:

$$\frac{\theta_3(2Q(\tau)|2\tau)}{\theta_2(2Q(\tau)|2\tau)} = \frac{Z_0^D(\tau)}{Z_{1/2}^D(\tau)}$$

This is an explicit transcendental equation on $Q(\tau)$.

- In the autonomous case $(2\pi i)^2 \frac{d^2Q(t)}{dt^2} = m^2 \wp'(2Q(t)|\tau_0)$ we have

$$\frac{\theta_2(2Q(t)|2\tau_0)}{\theta_3(2Q(t)|2\tau_0)} = \frac{\theta_2(2\omega t + \phi_0)|2\tau_{SW})}{\theta_3(2\omega t + \phi_0)|2\tau_{SW})}.$$

At this moment τ_{SW}, ω and ϕ_0 are just some functions of m and of the initial data.

Autonomous limit and $\mathcal{N} = 2^*$ theory

- To switch to autonomous case one needs to consider such big energies that period of the periodic motion is much smaller than 1 and τ:

$$H_\tau = (2\pi i \partial_\tau Q)^2 - m^2(\wp(2Q|\tau) + 2\eta_1(\tau)),$$

$\hbar \to 0$: $H_\tau = \hbar^{-2}(u + O(\hbar))$, $m = \hbar^{-1}\mu$, $\tau = \tau_0 + \hbar t$.

- Seiberg-Witten curve and energy conservation law are the same:

$$0 = \det(L - \lambda \mathbb{I}_2) \simeq \frac{1}{\hbar^2}\left[u + \mu^2\left(\wp(2z) + 2\eta_1(\tau)\right) + \tilde{\lambda}^2\right],$$

$$0 = u + \mu^2(\wp(2Q|\tau_0) + 2\eta_1(\tau_0)) + (2\pi\partial_t Q)^2$$

- It is always in the autonomous limit that $\det(A(z) - \lambda dz \cdot \mathbb{I}) = 0 \to$ Seiberg-Witten curve, the spectral curve of corresponding autonomous integrable system.

Autonomous limit and $\mathcal{N} = 2^*$ theory

- Computing the integral explicitly we get the map of parameters:

$$\tilde{u} = u - 4\pi i \mu^2 \partial_{\tau_0} \log \theta_2(\tau_0)$$

τ_0 is UV coupling, τ_{SW} is IR coupling:

$$\frac{\theta_2(2\tau_{SW})^2}{\theta_3(2\tau_{SW})^2} + \frac{\theta_3(2\tau_{SW})^2}{\theta_2(2\tau_{SW})^2} = \frac{\theta_2(2\tau_0)^2}{\theta_3(2\tau_0)^2} + \frac{\theta_3(2\tau_0)^2}{\theta_2(2\tau_0)^2} - \frac{\mu^2}{\tilde{u}} \frac{\pi^2 \theta_4(2\tau_0)^8}{\theta_2(2\tau_0)^2 \theta_3(2\tau_0)^2}$$

frequency:

$$\frac{\sqrt{\tilde{u}}}{2\pi i} \frac{\theta_2(2\tau_0)\theta_3(2\tau_0)}{\theta_2(2\tau_{SW})\theta_3(2\tau_{SW})}$$

Generalization to SL_N case, torus with many punctures

1. Fermionization formulas also exist, or we can even start directly from fermions [PG, Marshakov]
2. Formula for the kernel is

$$K(z_0, z) = Y^{-1}(z_0) x(\rho - \boldsymbol{Q}, z - z_0) Y(z)$$

3. The main relation:

$$Z^D(\tau, \rho) = \eta(\tau)^{-N} \prod_{i=1}^{N} \theta_1(\rho - Q_i(\tau)) \mathcal{T}(\tau)$$

4. Solution of a (part of) non-autonomous system — zeroes of $Z^D(\tau, \rho)$. Exactly like in Krichever's construction: $Z^D(\tau, Q_i) = 0$.
5. Decoupling of the $U(1)$ factor:

$$Z^D_{j/N}(\tau) = \Theta_j(\boldsymbol{Q}) \mathcal{T}(\tau)$$

Fredholm determinant

There exists Fredholm determinant for Z^D, analog of [PG, Lisovyy]:

$$Z^D(\tau) = q^{a^2-1/12} e^{2\pi i(\rho+1/2)} \det\left(\mathbb{I}+K\right), \qquad K = \begin{pmatrix} \mathsf{c} & \mathsf{d} \\ \mathsf{a} & \mathsf{b} \end{pmatrix}$$

$$\mathsf{a}(w,w') = D\frac{Y_0(qw)^{-1}Y_0(w') - \mathbb{I}}{qw-w'}, \quad \mathsf{b}(w,w') = -D\frac{Y_0(qw)^{-1}Y_\infty(w')}{qw-w'},$$

$$\mathsf{c}(w,w') = D^{-1}\frac{Y_\infty(\frac{w}{q})^{-1}Y_0(w')}{w/q-w'}, \qquad \mathsf{d}(w,w') = D^{-1}\frac{\mathbb{I} - Y_\infty(\frac{w}{q})^{-1}Y_\infty(w')}{\frac{w}{q}-w'},$$

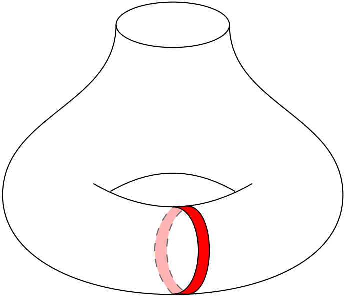
$$Y_0(w) = (1-w)^{(m-\gamma)}\times$$

$$\times\begin{pmatrix} {}_2F_1(m, m+2a, 2a, w) & \frac{-mw}{2a-1}{}_2F_1(1+m, 1+m-2a, 2-2a, w) \\ \frac{m}{2a}{}_2F_1(1+m, m+2a, 1+2a, w) & {}_2F_1(m, 1+m-2a, 1-2a, w) \end{pmatrix}$$

$$Y_\infty(w) = \sigma^x Y_0(1/w)\sigma^x, \quad D = -qe^{2\pi i\rho}\mathrm{diag}(q^{-a}e^{-2\pi i\beta}, q^a e^{2\pi i\beta})$$

Fredholm determinant

Construction of the determinant is related to the pants decomposition:



Y_0 and Y_∞ are two expansions of solution of the 3-point problem.

Fredholm determinant

One of the ways of computation of the determinant is expansion in a Laurent basis:

$$\mathrm{a}(w, w') = \sum_{p,q\in\mathbb{Z}'_-} \frac{\mathrm{a}_{pq}}{w^{p+\frac{1}{2}} w'^{q+\frac{1}{2}}}, \qquad \mathrm{b}(w, w') = \sum_{p\in\mathbb{Z}'_-, q\in\mathbb{Z}'_+} \frac{\mathrm{b}_{pq}}{w^{p+\frac{1}{2}} w'^{q+\frac{1}{2}}}$$

$$\mathrm{c}(w, w') = \sum_{p\in\mathbb{Z}'_+, q\in\mathbb{Z}'_-} \frac{\mathrm{c}_{pq}}{w^{p+\frac{1}{2}} w'^{q+\frac{1}{2}}}, \qquad \mathrm{d}(w, w') = \sum_{p,q\in\mathbb{Z}'_+} \frac{\mathrm{d}_{pq}}{w^{p+\frac{1}{2}} w'^{q+\frac{1}{2}}},$$

where $\mathbb{Z}'_+ = \{\frac{1}{2}, \frac{3}{2}, \frac{5}{3}, \ldots\}$, $\mathbb{Z}'_- = \{-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots\}$.

Mathematical proof will appear: [Del Monte, Desiraju, PG]

Thank you for your attention!