Supersymmetric Wilson Loops and Deformed $\mathcal{W}$-Algebras

(joint work with N. Haouzi, arXiv:1906.xxxxx)

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Wilson Loops

Wilson loops gauge invariant non-local operators encoding important aspects of the strongly-coupled regime of gauge theories. They are defined as the trace of the holonomy matrix, evaluated in some representation *R* of the gauge group

We are interested in studying supersymmetric Wilson loops defined as

$$S^{1d} = \int dt\, \chi^\dagger \, (\partial_t - i\, A_t + \Phi + \, M)\, \chi$$

where A_t and Φ are the pullback of the 5d gauge field and the adjoint scalar of the vector multiplet, respectively. *M* is the real mass, and χ is a 1d fermion field.

Wilson Loops

We would like to compute the path integral with the insertion of the Wilson loop

$$[\chi^{\mathfrak{g}}]^{5d} = \int D\psi D\chi \, e^{i(S^{5d}[\psi]+S^{1d}[\psi,\chi,M])} = z^{-n/2} \sum_{j=0}^{n} (-z)^j \langle W_{\Lambda^j} \rangle$$

It has been shown that the path integral computes the generating function of Wilson loops on anti-symmetric representations, where $z \equiv e^M$.

The expectation value is still a polynomial in the instanton background!

Wilson Loops

The presence of Wilson lines generalizes the ADHM construction, since when an instanton moves in the presence of a quark, it now also experiences a Lorentz force. The endeavor of understanding the dynamics of instantons in this modified background was initiated only recently.

Our aim is to generalize these results to 5d quivers described by simple Lie groups!

Type IIB: Little strings

Little String Theory

Little String Theory (LST) is a *6d* string theory with no coupling to gravity. Type IIB string theory compactified on an ADE surface X:

$$X = \widetilde{\mathbb{C}^2/\Gamma_{\mathfrak{g}}}$$

where $\Gamma_{\mathfrak{g}}$ is discrete sub-group of $SU(2)$ related to the (simply-laced) group $\mathfrak{g}$.

Bulk modes of type IIB(A) superstring theory are decoupled by taking the limit:

$$g_s \to 0$$

Taking further the limit $m_s \to \infty$ ($\ell_s \to 0$), one gets (2,0) *6d* SCFT labeled by $\mathfrak{g}$.

Little String Theory

LST contains a non-abelian self-dual tensor field, and its moduli space is

$$\mathcal{M} = (\mathbb{R}^4 \times S^1)^n / W$$

where n is the rank, and W is the Weyl group of $\mathfrak{g}$. The moduli are encoded as periods of triplet of self-dual two-forms, and NS and RR B-fields:

$$\frac{m_s^4}{g_s} \int_{S_a} \omega_{I,J,K}, \qquad \frac{m_s^2}{g_s} \int_{S_a} B_{NS}, \qquad m_s^2 \int_{S_a} B_{RR},$$

where two-cycles S_a generate $H_2(X, \mathbb{Z})$. In the $g_s \to 0$ limit, we keep these parameters *fixed*.

Little String Theory

Co-dimension two defects in LST, using D5 branes wrapped on two cycles on the following type IIB background with ADE singularities are studied by Aganagic and Haouzi:

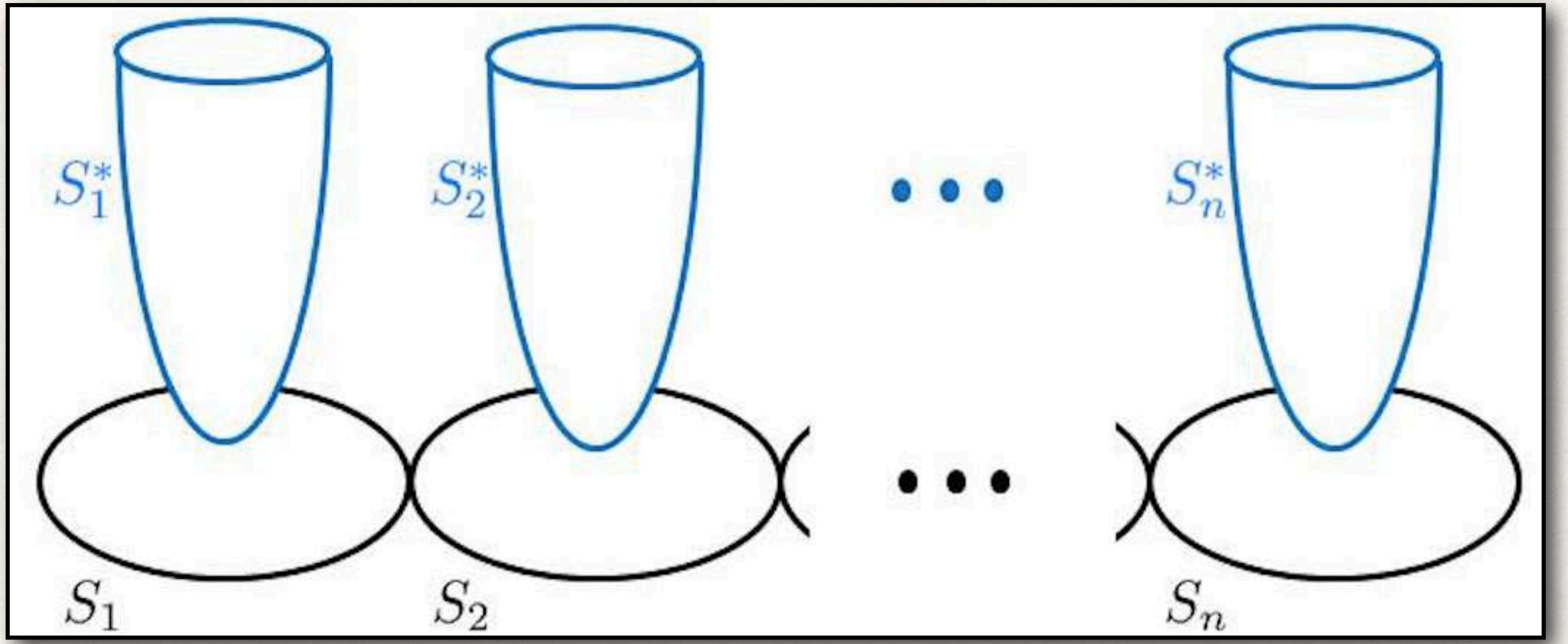
$$\begin{array}{rccccc} \text{type IIB:} & \mathcal{C} & \times & \mathbb{C}^2 & \times & X \\ & \Cup & & & & \cup \\ \text{D5:} & \{p_i\} & \times & \mathbb{C}^2 & \times & S_a \text{ or } S_a^* \end{array}$$

where we take $\mathcal{C}$ to be a cylinder; since $\mathcal{C}$ is flat, this is a type IIB background. S_a^* are generators of the relative homology group $H_2(X, \partial X, \mathbb{Z})$.

$\{S_a\}$ and $\{S_a^*\}$ form bases for compact and non-compact two cycles, respectively.

Little String Theory

One can visualize the cycles like in the following picture:



Picture from Aganagic & Haouzi

$$\#(S_a \cap S_b^*) = \delta_{ab}$$

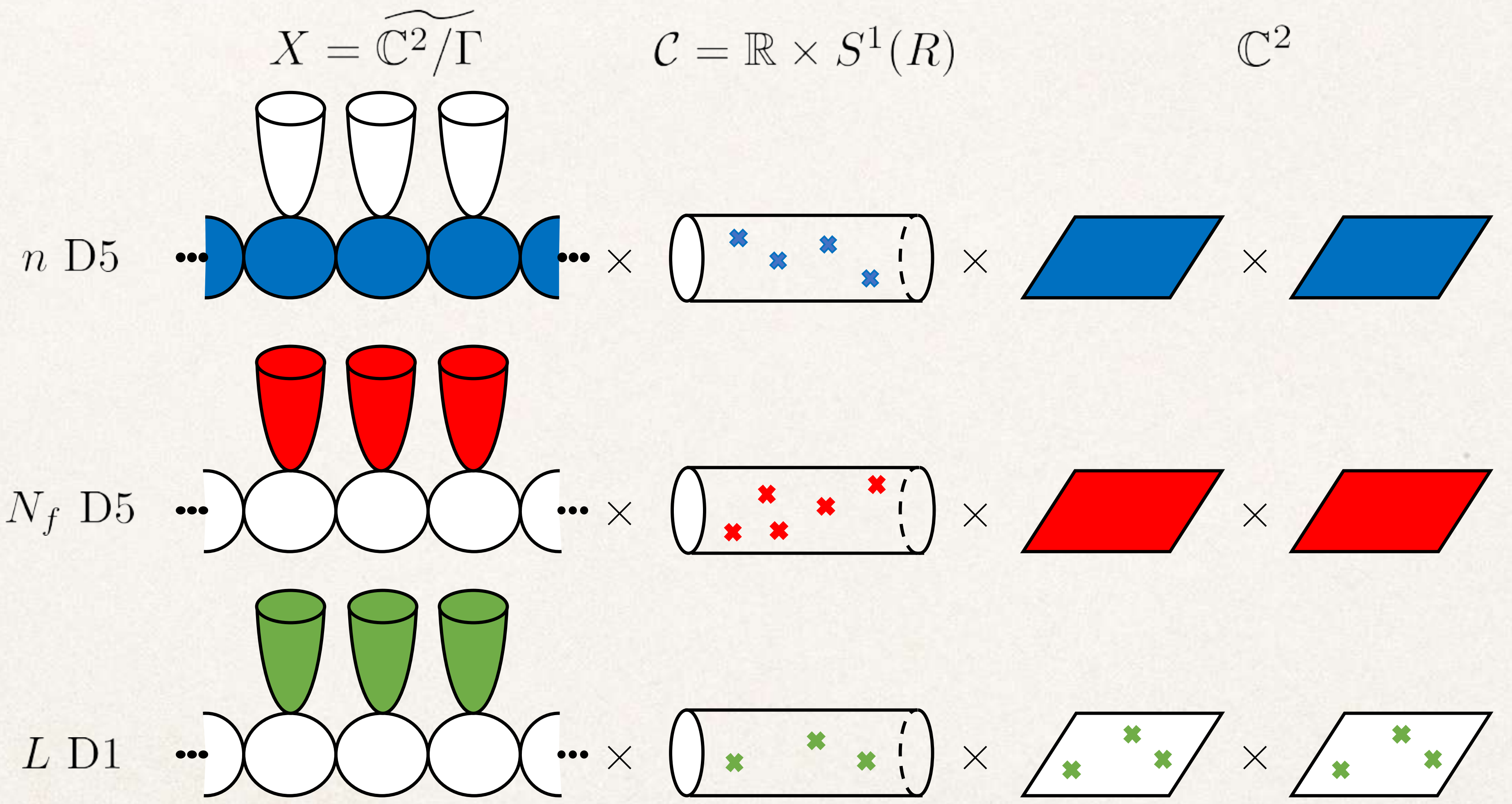
Little String Theory

According to the McKay correspondence, the representation and geometry theory associated to $\mathfrak{g}$ are ultimately connected:

Cartan matrix of group $\mathfrak{g}$	$\longleftrightarrow$	Intersection matrix of compact two cycles S_a *(up to a sign)*
Roots in the root lattice Λ	$\longleftrightarrow$	Compact two cycles in $H_2(X,\mathbb{Z})$ $[S] = \sum_{a=1}^{n} d_a\, \alpha_a \in \Lambda$
Weights in weight lattice Λ_*	$\longleftrightarrow$	Non-compact two cycles in $H_2(X,\partial X,\mathbb{Z})$ $[S^*] = -\sum_{a=1}^{n} m_a\, w_a \in \Lambda_*$

5d Gauge Theory (ADE)

We need to wrap branes to study 5d gauge theory with Wilson loops insertions:



5d Gauge Theory (ADE)

We are not interested in arbitrary brane configurations, but focus on the one leading to conformal theories (in 4d sense). The total brane flux should vanish:

$$[S + S^*] = 0 \qquad \Longleftrightarrow \qquad \sum_{b=1}^{n} C_{ab}\, d_b = m_a$$

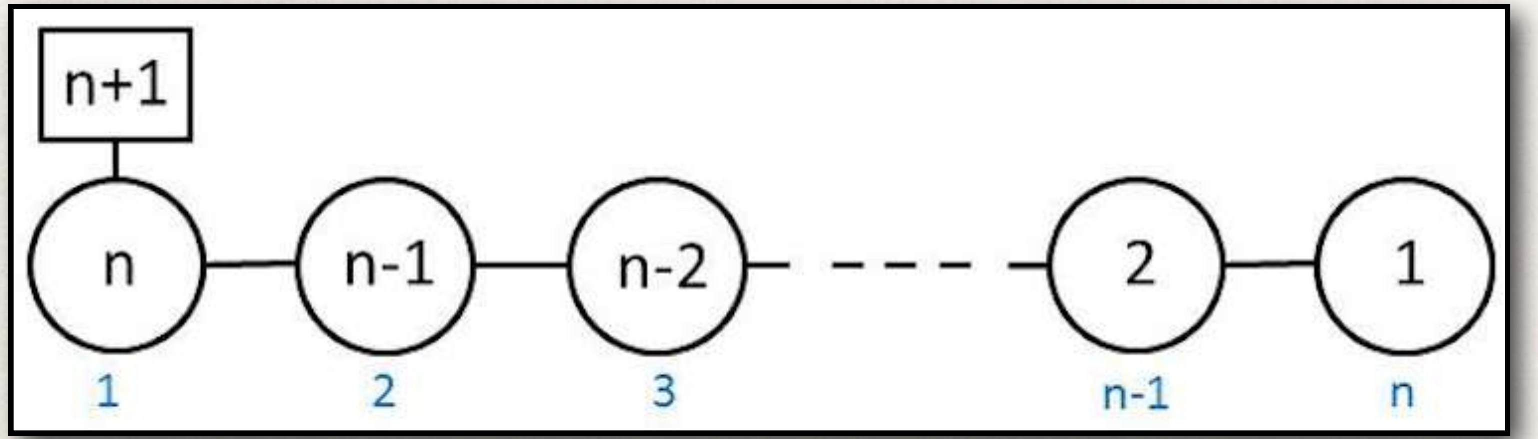
For a quiver gauge theory description, the branes wrapping different components should preserve the same supersymmetry. A natural choice of the moduli of LST is given by:

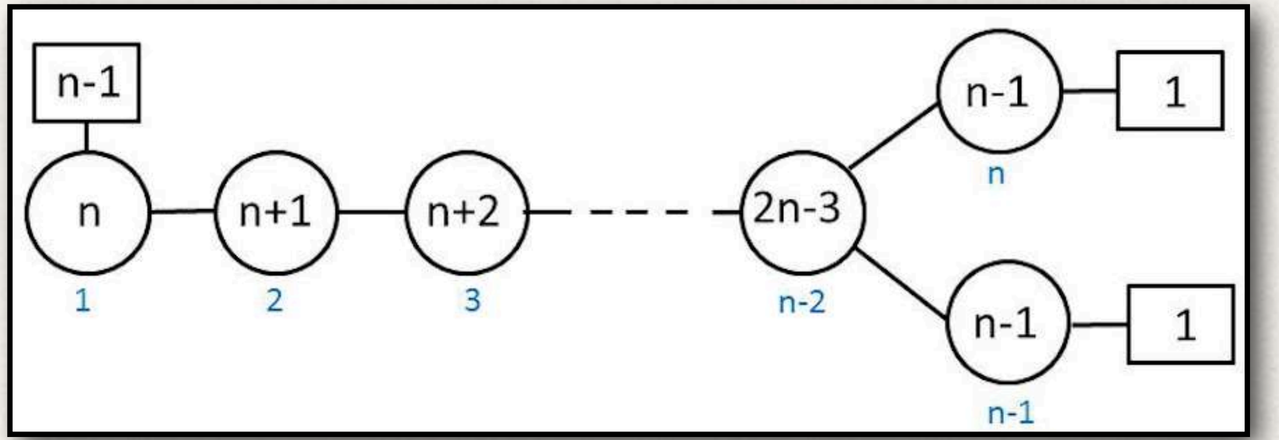
$$\int_{S_a} \omega_{J,K} = 0, \; \int_{S_a} B_{NS} = 0, \qquad \tau_a = \int_{S_a} (m_s^2 \omega_i / g_s + i B_{RR}),$$

where τ_a is identified with the complexified gauge coupling constant of the a^{th} node.

5d Gauge Theory (ADE)

Due to the circle on the cylinder $\mathcal{C}$, the gauge theory is not a 4d, but a 5d theory on a circle.



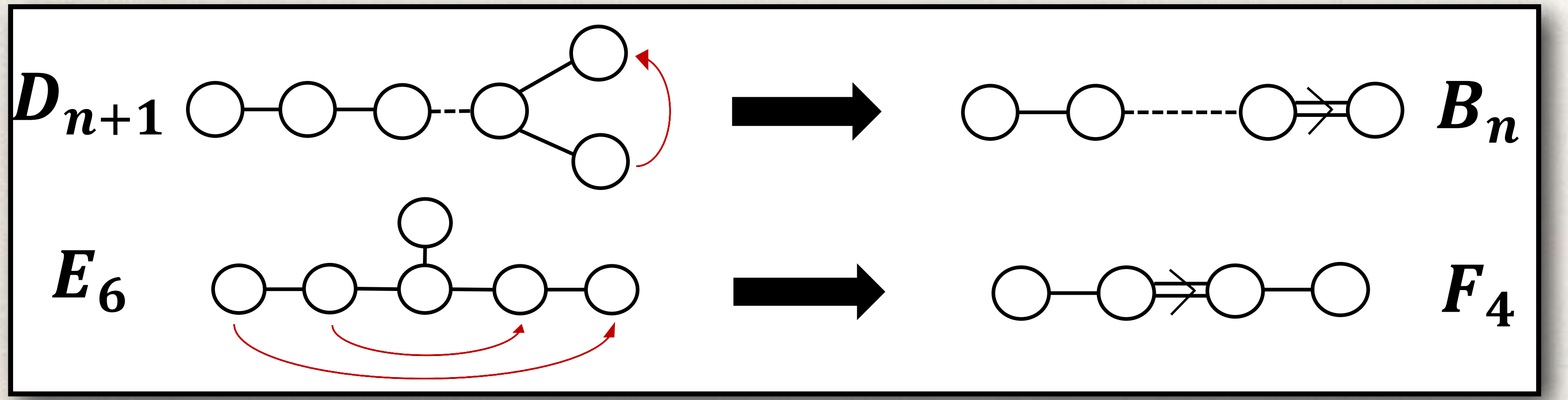


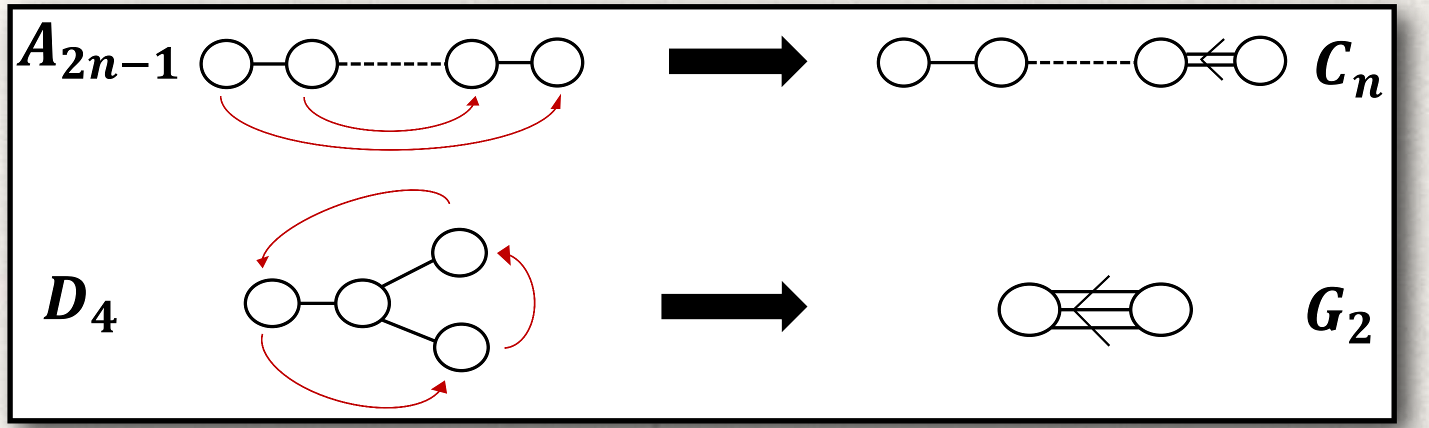
D5 branes are points on $\mathcal{C}$, the positions of branes wrapping compact cycles encode the Coulomb moduli, and the positions of branes on non-compact cycles are masses.

Triality emerges when we probe the root of the Higgs branch, when the positions of all compact branes coincide with the position of at least one non-compact brane.

5d Gauge Theory (ABCDEFG)

There is no space X with B, C, F or G singularity that we can use to study LST for the non-simply laced groups! However, we can use the outer automorphism on simply-laced groups to fiber ADE type singularity over $\mathcal{C} \times \mathbb{C}^2$.





If we go around the origin of one of the complex planes $\mathbb{C}$, X goes back to itself up to the outer automorphism. We will come back to this point!

5d Gauge Theory (ABCDEFG)

According to this action, we group the nodes as long (roots that are invariant) and short roots. Therefore, we need to take into account the relative lengths of the nodes!

We need to distinguish between the root (weight) and co-root (co-weight) lattices for non-simply laced groups.

D5 branes wrapping non-compact cycles are decomposed now in terms of fundamental coweights:

$$[S^*] = -\sum_{a=1}^{n} m_a \, w_a^\vee \quad \in \Lambda_*^\vee$$

D5 branes wrapping compact cycles are decomposed now in terms of fundamental coroots:

$$[S] = \sum_{a=1}^{n} d_a \, \alpha_a^\vee \quad \in \Lambda^\vee$$

5d Gauge Theory (ABCDEFG)

Example: Sphere with 3 full F_4 punctures. Let us pick $\mathcal{W} = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$

Dynkin labels

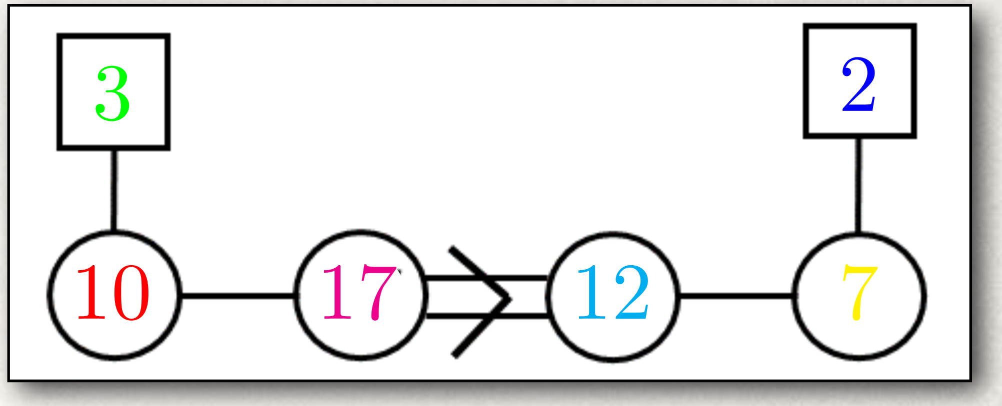
$$\omega_1 = [\ \ 0,\ \ 0,\ \ 0,\ \ 1]$$
$$\omega_2 = [\ \ 0,\ \ 0,\ \ 1,-1]$$
$$\omega_3 = [\ \ 0,\ \ 1,-1,\ \ 0]$$
$$\omega_4 = [\ \ 1,-1,\ \ 0,\ \ 0]$$
$$\omega_5 = [-1,\ \ 0,\ \ 0,\ \ 0]$$

$$\sum_{i=1}^{5} \omega_i = 0$$

Expanded in fund. co-weights

$$\omega_1 = -w_4^\vee + 4\alpha_1^\vee + 8\alpha_2^\vee + 6\alpha_3^\vee + 4\alpha_4^\vee$$
$$\omega_2 = -w_4^\vee + 4\alpha_1^\vee + 8\alpha_2^\vee + 6\alpha_3^\vee + 3\alpha_4^\vee$$
$$\omega_3 = -w_1^\vee + \alpha_1^\vee + \alpha_2^\vee$$
$$\omega_4 = -w_1^\vee + \alpha_1^\vee$$
$$\omega_5 = -w_1^\vee$$

5d quiver



5d Gauge Theory

We want to compute the K-theoretic partition function $[\chi^{\mathfrak{g}}]^{5d}$ with respect to:

$$(\mathbb{C}^*)^2 \times \prod_{a=1}^{m} U(d_a) \times \prod_{a=1}^{m} U(m_a) \times \prod_{a=1}^{m} U(M_a)$$

The partition function takes the following compact form:

$$[\chi^{\mathfrak{g}}]^{5d}_{(L^{(1)},\dots,L^{(m)})}(z_\rho^{(a)}) = \sum_{\omega\in V(\lambda^\vee)} \prod_{b=1}^{m} \left(\widetilde{\mathfrak{q}}^{(b)}\right)^{d_b^\omega} c_{d_b^\omega}(q,t) \left(Q^{(b)}(z_{*\rho}^{(a)})\right)^{d_b^\omega} \left[\widetilde{Y}_{5d}(z_\rho^{(a)})\right]_\omega$$

It is a character of finite dimensional irreducible representation $V(\lambda^\vee)$ of $U_q(\widehat{\mathfrak{g}})$ with highest co-weight $\lambda^\vee = \sum_{a=1}^{m} L^{(a)}\,\lambda_a^\vee$

5d Gauge Theory

In the case of ADE, Nekrasov called $[\chi^{\mathfrak{g}}]^{5d}$ a *qq-character.*

$$\left[\chi^{A_1}\right]^{5d} = \langle Y(z)\rangle + \tilde{\mathfrak{q}} \left\langle \frac{Q(z\sqrt{t/q})}{Y(z\,t/q)} \right\rangle$$

$$\left[\chi^{A_2}\right]^{5d}_1 = \langle Y_1(z)\rangle + \tilde{\mathfrak{q}}_1 \left\langle \frac{Q_1(z\sqrt{t/q})\,Y_2(z\,\mu)}{Y_1(z\,t/q)} \right\rangle + \tilde{\mathfrak{q}}_1\tilde{\mathfrak{q}}_2 \left\langle \frac{Q_1(z\sqrt{t/q})\,Q_2(z\,\mu\sqrt{t/q})}{Y_2(z\,\mu t/q)} \right\rangle$$

$$\left[\chi^{A_2}\right]^{5d}_2 = \langle Y_2(z)\rangle + \tilde{\mathfrak{q}}_2 \left\langle \frac{Q_2(z\sqrt{t/q})\,Y_1(z\,\mu^{-1}t/q)}{Y_2(z\,t/q)} \right\rangle + \tilde{\mathfrak{q}}_2\tilde{\mathfrak{q}}_1 \left\langle \frac{Q_2(z\,t/q)\,Q_1(z\,\mu^{-1}\sqrt{t/q}^{\,3})}{Y_1(z\,\mu^{-1}t^2/q^2)} \right\rangle$$

Fractional Quivers

Fractional Quivers

We engineered the non-simply laced quivers by fibering the ADE type singularity over $\mathcal{C} \times \mathbb{C}^2$, and considered if we go around one of complex planes, $\mathbb{C}$. Kimura and Pestun studied the instanton partition function for such quivers, called *fractional quivers,* using localization.

Losev, Moore, Nekrasov and Shatashvili introduced the deformed 5d background, $\mathbb{C}^2 \times S^1$:

$$z_1 \mapsto e^{i\epsilon_1} z_1 \equiv q\, z_1, \qquad z_2 \mapsto e^{i\epsilon_2} z_2 \equiv t^{-1}\, z_2$$

This action is modified for fractional quivers by the relative length of the roots:

$$z_1 \mapsto e^{i r_a \epsilon_1} z_1 \equiv q^{r_a}\, z_1, \qquad z_2 \mapsto e^{i\epsilon_2} z_2 \equiv t^{-1}\, z_2$$

Fractional Quivers

The instanton partition function can again be written as an index, but modified due to the action of the outer automorphism group:

$$Z_{5d}\left(\mathbb{C}^2 \times S^1\right) = \mathrm{tr}\left(-1\right)^F q^{r_a(S_1 - S_R)} t^{-S_2 + S_R},$$

where $S_{1,2}$ and S_R are generators of two rotations around the complex planes and R-symmetry, respectively. The generating function can again be written as a sum over fixed points:

$$Z_{5d}(\mathbb{C}^2 \times S^1) = r_{5d} \sum_{\{\mu\}} I_{5d,\{\mu\}}(q, t; a, m, \tau)$$

with modified contributions to different multiplets!

Fractional Quivers

Let us define the generalized Nekrasov factor:

$$N_{\mu^a \mu^b}(Q; q^{r_{ab}}) = \prod_{i,j=1}^{\infty} \frac{\left(Q q^{r_a \mu_i^a - r_b \mu_j^b} t^{j-i+1}; q^{r_{ab}}\right)_\infty}{\left(Q q^{r_a \mu_i^a - r_b \mu_j^b} t^{j-i}; q^{r_{ab}}\right)_\infty} \frac{\left(Q t^{j-i}; q^{r_{ab}}\right)_\infty}{\left(Q t^{j-i+1}; q^{r_{ab}}\right)_\infty}$$

Vector multiplets

$$z^{5d}_{V_a, \vec{\mu}^a} = \prod_{1 \leq I, J \leq d_a} [N_{\mu_I^a \mu_J^a}(e_{a,I}/e_{a,J}; q^{r_a})]^{-1}$$

Bifundamental hypers

$$z^{5d}_{H_{ab}, \vec{\mu}^a, \vec{\mu}^b} = \prod_{1 \leq I \leq d_a} \prod_{1 \leq J \leq d_b} [N_{\mu_I^a \mu_J^b}(e_{a,I}/e_{b,J}; q^{r_{ab}})]^{\Delta_{ab}}$$

$$r_{ab} = \gcd(r_a, r_b)$$

Fundamental hypers

$$z^{5d}_{H_a, \vec{\mu}^a} = \prod_{1 \leq \alpha \leq m_a} \prod_{1 \leq I \leq d_a} N_{\emptyset \mu_I^a}(v_a^2 f_{a,\alpha}/e_{a,I}; q^{r_a})$$

$$v_a \equiv \sqrt{q^{r_a}/t}$$

Deformed $\mathcal{W}$-algebras: $\mathcal{W}_{q,t}(\mathfrak{g})$

Deformed $\mathcal{W}$-algebras: $\mathcal{W}_{q,t}(\mathfrak{g})$

$\mathfrak{g}$-type Toda field theory can be written in terms of free bosons in 2d with background charge and exponential potential:

$$S_{Toda} = \int dz d\bar{z} \, \sqrt{g} \, g^{z\bar{z}} [\langle \partial_z \varphi, \partial_{\bar{z}} \varphi \rangle + \langle \rho, \varphi \rangle \, QR + \sum_{a=1}^{n} e^{\langle \alpha_a^\vee, \varphi \rangle / b}]$$

The conformal blocks of the Toda CFT in free field formalism takes the form,

$$\langle V_{\beta_1}^\vee(z_1) \ldots V_{\beta_k}^\vee(z_k) \quad \prod_{a=1}^{n} (Q_a^\vee)^{N_a} \rangle_{free} \, .$$

where the primary vertex operators are defined as

$$V_\beta^\vee(z) = e^{\langle \beta, \varphi(z) \rangle}$$

Deformed $\mathcal{W}$-algebras: $\mathcal{W}_{q,t}(\mathfrak{g})$

The screening charges are defined as integrals of screening current operators:

$$Q_a^\vee \equiv \oint dx\, S_a^\vee(x), \quad \text{with } S_a^\vee(z) = e^{\langle \alpha_a^\vee, \phi(z) \rangle / b}\,.$$

Frenkel and Reshetikhin introduced a quantum deformation of the $\mathcal{W}(\mathfrak{g})$ algebra by deforming free field representation. Introduce the quantum number:

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

The symmetrized Cartan matrix is given by

$$B_{ab}(q,t) = [r_a]_q \left(q^{r_a} t^{-1} + q^{-r_a} t\right) \delta_{ab} - [I_{ab}]_q$$

Deformed $\mathcal{W}$-algebras: $\mathcal{W}_{q,t}(\mathfrak{g})$

They construct q-deformed Heisenberg algebra generated by simple root generators:

$$[\alpha_a[k],\alpha_b[m]]=\frac{1}{k}(q^{\frac{k}{2}}-q^{-\frac{k}{2}})(t^{\frac{k}{2}}-t^{-\frac{k}{2}})B_{ab}(q^{\frac{k}{2}},t^{\frac{k}{2}})\delta_{k,-m}$$

The screening current operators can be explicitly written as

$$S_a^\vee(x)=x^{-\alpha_a[0]/r_a}\ :\exp\Big(\sum_{k\neq 0}\frac{\alpha_a[k]}{q^{\frac{k\,r_a}{2}}-q^{-\frac{k\,r_a}{2}}}\,e^{kx}\Big):$$

The fundamental weight generators $w_a[m]$ satisfy the commutation relation:

$$[\alpha_a[k],w_b[m]]=\frac{1}{k}(q^{\frac{k\,r_a}{2}}-q^{-\frac{k\,r_a}{2}})(t^{\frac{k}{2}}-t^{-\frac{k}{2}})\,\delta_{ab}\,\delta_{k,-m}$$

Deformed $\mathcal{W}$-algebras: $\mathcal{W}_{q,t}(\mathfrak{g})$

The vertex operators are defined in terms of the weight generators $w_a[m]$:

$$V_a^\vee(x) = x^{w_a[0]/r_a} \; : \exp\Big(-\sum_{k\neq 0} \frac{w_a[k]}{q^{\frac{k\, r_a}{2}} - q^{-\frac{k\, r_a}{2}}}\, e^{kx}\Big) :$$

For a given node on the quiver, different two point functions give different multiplet contributions:

$$\langle S_a^\vee(x)\, S_a^\vee(x')\rangle_{free} = \frac{(e^{x-x'};q^{r_a})_\infty}{(t\, e^{x-x'};q^{r_a})_\infty}\, \frac{(e^{x'-x};q^{r_a})_\infty}{(t\, e^{x'-x};q^{r_a})_\infty}\, \frac{\theta_{q^{r_a}}(t\, e^{x-x'})}{\theta_{q^{r_a}}(e^{x-x'})}$$

vector multiplet contr. (in 3d)

$$\langle S_a^\vee(x)\, V_b^\vee(x')\rangle_{free} = \frac{(t\, v_a\, e^{x'-x};q^{r_a})_\infty}{(v_a\, e^{x'-x};q^{r_a})_\infty}$$

fundamental multiplet contr. (in 3d)

Deformed $\mathcal{W}$-algebras: $\mathcal{W}_{q,t}(\mathfrak{g})$

Two point functions of screening currents between nodes connected by a link on the Dynkin diagram give bifundamental contributions:

$$\langle S_a^\vee(x)\, S_b^\vee(x')\rangle_{free} = \frac{(t\, v_{ab}\, e^{x-x'}; q^{r_{ab}})_\infty}{(v_{ab}\, e^{x-x'}; q^{r_{ab}})_\infty}$$

Deformed $\mathcal{W}$-algebras: $\mathcal{W}_{q,t}(\mathfrak{g})$

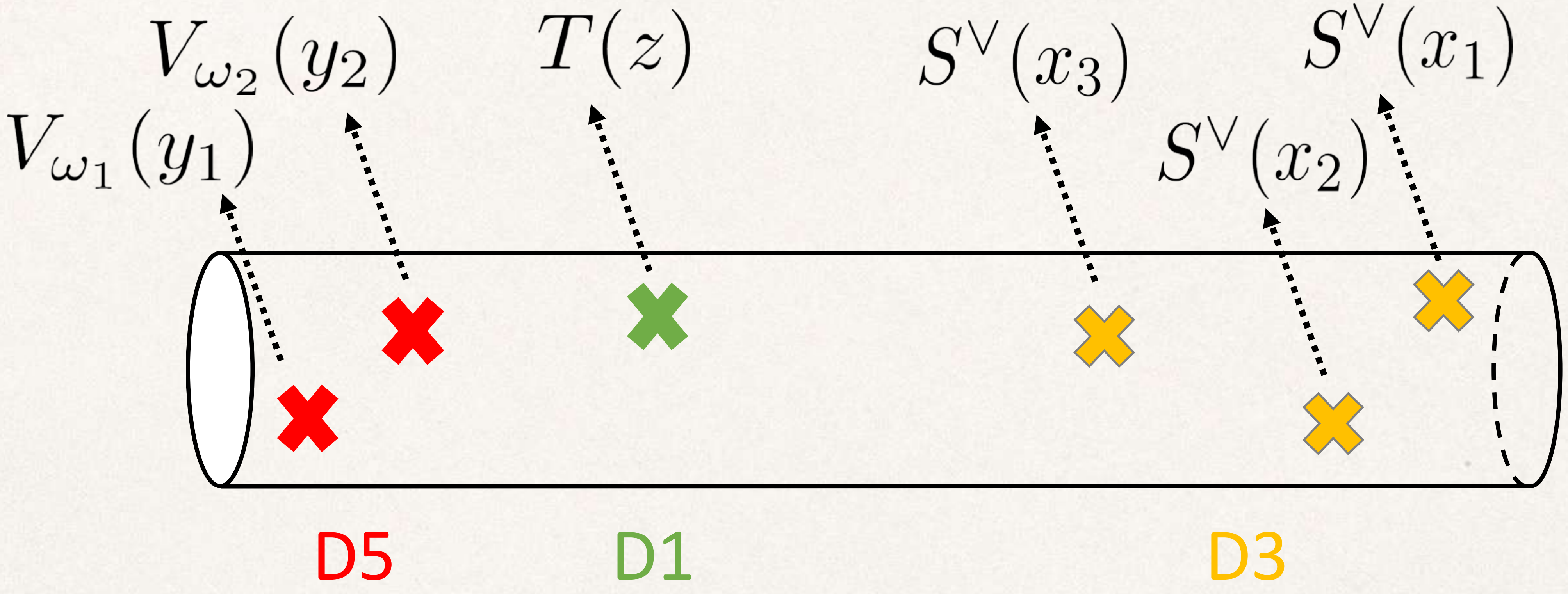
We can understand the Wilson loops also from deformed $\mathcal{W}$-algebra side by inserting the energy-momentum tensor and higher spin fields. q-deformation of them are defined by Frenkel and Reshetikhin:

For example: A_ℓ

$$\Lambda_i(z) = :Y_i(zq^{-(i-1)/2}t^{(i-1)/2})Y_{i-1}(zq^{-i/2}t^{i/2})^{-1}:, \qquad i = 1, \ldots, \ell+1$$

$$T(z) = \sum_{i=1}^{\ell+1} \Lambda_i(z)$$

Deformed $\mathcal{W}$-algebras: $\mathcal{W}_{q,t}(\mathfrak{g})$



Gauge/Liouville Triality

Triality

The triality is shown to exist 5d instanton partition function, 3d vortex partition function and *q*-deformed Toda theory when we probe the root of the Higgs branch:

$$e_{a,I} = f_i \, t^{N_{a,I}} v^{\#_{a,i,I}} q^{\#'_{a,i,I}}$$

At this point, the 5d instanton partition functions becomes 3d vortex partition function evaluated by residues:

$$Z_{3d} = \int dx \, I_{3d}(x) = \sum_{\{\mu\}} \mathrm{res}_{\{\mu\}} \, I_{3d}(x)$$

The integral is identical to the Dotsenko-Fateev representation of the Toda conformal blocks!

Triality

More precisely, we have following simplification:

$$\prod_{a=1}^{n} z^{5d}_{V_a,\mu^a} = \prod_{a=1}^{n} \frac{z^{3d}_{V_a}(x_{\mu^a})}{z^{3d}_{V_a}(x_\varnothing)} \cdot V_{vect}$$

$$\prod_{a<b} z^{5d}_{H_{ab},\mu^a,\mu^b} = \prod_{a<b} \left[\frac{z^{3d}_{H_{ab}}(x_{\mu^a}, x_{\mu^b})}{z^{3d}_{H_{ab}}(x_\varnothing, x_\varnothing)}\right]^{\Delta_{ab}} \cdot V_{bifund}$$

$$\prod_{a=1}^{n} z^{5d}_{H_a,\mu^a} = V_{fund}$$

$$\prod_{a=1}^{n} z^{5d}_{CS,\mu^a} = V_{CS}$$

$$\Bigg\} \; V_{vect}V_{bifund}V_{fund}V_{CS} = \prod_{a=1}^{n} \frac{z^{3d}_{H_a}(x_{\mu^a})}{z^{3d}_{H_a}(x_\varnothing)}$$

Thank you