From 3d dualities to 2d free field correlators and back

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based on 1712.08140 with Nedelin and Zenkevich,
1812.08142 with Aprile and Zenkevich
and 1903.10817, 1905.05807 with Sacchi

Dualities

Describing strongly interacting QFTs is very challenging. We expect/hope that strongly coupled phases could be described by a new set of weakly coupled emergent degrees of freedom.

In presence of extra symmetries sometimes it is possible to realise this paradigm in the context of dualities. Two main examples:

- IR dualities: we can describe the strongly coupled phase of a gauge theory in terms of a different weakly coupled gauge theory.
- Holographic dualities: the emergent fields live in a space with one extra dimension and the dual theory is a gravity theory.

IR dualities

First example for $4d$ $\mathcal{N} = 1$ theories, Seiberg '94:

eSQCD: $SU(N_c)$ with N_f flavor has a dual description as mSQCD: $SU(N_f - N_c)$ with N_f flavors and extra singlets Φ_{ij} with $\mathcal{W} = \Phi_{ij} q_i \tilde{q}_j$.

In the confining region $N_c + 1 < N_f \leq 3/2 N_c$ the weakly coupled dof are those of the magnetic dual mSQCD which is IR free there.

In the conformal window $3/2 N_c < N_f < 3 N_c$ eSQCD and mSQCD flow to an IR SCFT.

The Seiberg duality has been generalized to other gauge groups and dimensions.

Testing IR dualities

Tests have to be non-perturbative, typical strategies:

- Anomaly matching
- Map the gauge invariant operators of the electric theories to those of the magnetic duals
- Consistency checks: integrate out some of the fields and see what happens in the magnetic dual
- Matching exact quantities computed via localisation, since 2007.

Localization industry

In 2007 Pestun applied Witten's localisation to the path integral of theories with 8 supercharges on the sphere S^4,
in 2009 Kapustin-Willett-Yaakov applied localisation on S^3. Many generalisation followed.

Localising a SUSY theory on a curved background is a two step process:

- I) Formulate the theory on the curved background preserving some SUSY.

  First works did so by adding extra $1/r$ terms to the action.

  Systematic approach: start from off-shell supergravity, which is later decoupled leaving behind a rigid susy theory with extra background fields initiated in [Festuccia-Seiberg'11].

- II) Calculate one-loop determinants $\rightarrow$ special functions!

The special functions feast

A surprising harmony emerges from the calculation of determinants.

For example for spheres with *squashing* parameters $\epsilon_1 \cdots \epsilon_r$ the localised partition functions are written in terms of two types of special functions:

$$\Upsilon_r(x|\epsilon) = \gamma_r\,(x|\epsilon)\,\gamma_r(\sum_i \epsilon_i - x|\epsilon)^{(-1)^r}\,, \quad S_r(x|\epsilon) = \gamma_r(x|\epsilon)\gamma_r(\sum_i \epsilon_i - x|\epsilon)^{(-1)^{r-1}}\ .$$

with

$$\gamma_r(x|\epsilon) = \prod_{n_1 \cdots n_r = 0}^{\infty} (x + n_1\epsilon_1 + \cdots + n_r\epsilon_r)\,.$$

For even and odd spheres a vector multiplet contributes as:

$$Z_{S^{2r}} = \int \prod_{i=1}^{N} da_i \prod_{i<j} \Upsilon_r(a_i - a_j|\epsilon)\ e^{P_r(a)} + \cdots\ ,$$

$$Z_{S^{2r-1}} = \int \prod_{i=1}^{N} da_i \prod_{i<j} S_r(a_i - a_j|\epsilon)\ e^{P_r(a)} + \cdots\ .$$

The degree r polynomials $P_r(a)$ come from the classical action (Yang-Mills, Chern-Simons, FI couplings).

Applications

- Many highly non-trivial tests of holographic dualities for 3d, 4d and 5d theories (large N matrix model techniques).
- Tests and new proposals for IR dualities in various dimensions (heavy use of mighty integral identities for special functions).
- Correspondences: establish a map between susy gauge theories and low dimensional integrable models. Offer the possibility of attacking problems with two totally different perspectives.

Correspondences

- Gauge/BPS initiated by [Nekrasov-Shatashvili'09]. Establish a map between 2d SUSY gauge theories and quantum integrable systems:
  - BPS vacua equations $\leftrightarrow$ Bethe equations
  - Instanton partition function $\rightarrow$ Quantization of the integrable system

- Gauge/CFT initiated by [Alday-Gaiotto-Tachikawa'09]. Establish a map between 4d SUSY gauge theories and 2d CFTs:
  - S^4 partition functions $\leftrightarrow$ Toda CFT correlators
  - Surface operators $\leftrightarrow$ degenerate primary operators insertions.
  - Wilson-'t Hooft loops $\leftarrow$ Verlinde loop operators.
  - gauge/free-field correspodence [Aganagic-Haouzi-Kozcaz-Shakirov]: $D^2 \times S^1$ partition functions of $3d$ $\mathcal{N} = 2$ quiver theories $\leftrightarrow$ free field Dotsenko-Fateev correlators in q-deformed Toda (ranks $\leftrightarrow$ screening charges, real masses $\leftrightarrow$ momenta and insertion points).

I made the point that IR dualities and correspondences are two very different things, but I will now try to show that actually they are somehow related.

$3d \to 2d$ reduction of $\mathcal{N} = 2$

Quite subtle! problems with on-compact branches, role of the metric...[Aharony-Razamat-Willett].

We focus on $D^2 \times S^1$ or $S^2 \times S^1$ partition functions of mass deformed theories, written in terms of $(x;q)_\infty = \prod_{k=1}^{\infty}(1 - xq^k)$, where $q \sim e^R$.

$2d$ limits depend on how the $3d$ real masses scale with R:

- Higgs or natural limit: matter fields remain light, we get a 2d gauge theory. At the level of partition functions

$$\lim_{q\to 1} \frac{(q^x;q)_\infty}{(q^y;q)_\infty} = (-\hbar)^{y-x}\,\frac{\Gamma(y)}{\Gamma(x)}\,,$$

  where S^2 and D^2 partition functions are written in terms of $\Gamma(x)$.
- Coulomb un-natural limit: matter fields are heavy, we get Landau-Ginsburg models of 2d twisted multiplets. At the level of partition functions

$$\lim_{q\to 1} \frac{(q^a x;q)_\infty}{(q^b x;q)_\infty} = (1-x)^{b-a}\,.$$

  However we can map these integrals to 2d free field DF correlators!

Reducing dual pairs

Reducing partition functions of dual theories, we find the following scenarios:

- Higgs limit of a 3d Seiberg-like duality identity yields a similar 2d Seiberg-like duality identity.
- Coulomb limit of a 3d Seiberg-like duality identity yields a duality identity between LG models or a duality identity between free field correlators.
- For a mirror or a spectral dual pair, Higgs limits on one side becomes Coulomb limit on the other side and we get a Hori-Vafa-like duality identity or a gauge/free field correspondence.

WARNING! The duality identities above do not necessarily represent 2d dualities! [Aharony-Razamat-Willett]

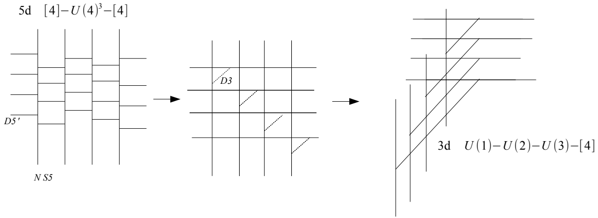
I will focus on the relation between mass deformed theories and free field correlators, 3d or 2d mass parameters are mapped to momenta and insertion points.

Let's start with how we recover gauge/free field correlators from 3d spectral dualities [Nedelin-SP-Zenkevich], [Aprile-SP-Zenkevich].

3d defect theories via Higgsing

A $U(N)^{N-1}$ $5d$ quiver is realised on a web of N NS5 and N D5' branes.

When the parameters are tuned to special values $\equiv$ Higgsing condition the NS5 branes can be removed from the web and D3 can be stretched.



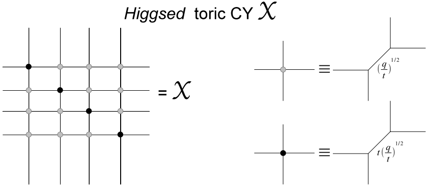
The 3d theory on the D3s is our defect theory $FT[SU(N)]$, which is $T[SU(N)]$ with an extra set of singlets and $\delta\mathcal{W} = Q_i\tilde{Q}_jX_{ij}$.

The 5d theory is invariant under S-duality (NS5 $\leftrightarrow$ D5') $\Rightarrow$ $FT[SU(N)]$ is invariant under 3d spectral duality.

Higgsing and geometric transition

The 5d quiver theory can be geometrically engineered in M theory on $X \times \mathbb{R}^4_{q,t} \times S^1$ where X is a toric Calabi-Yau.

The Higgsing conditions translate into quantisation conditions for the Kählers parameters of X. At these values there is geometric transition:



Using the refined topological vertex [Iqbal-Kozcaz-Vafa] we can calculate the partition function of the *Higgsed* CY $\mathcal{X}$ and compare with the localization result:

$$Z^{\mathcal{X}}_{\text{top}}(\vec{\mu}, \vec{\tau}; q, t) = \mathcal{B}^{D_2 \times S^1}_{FT[SU(N)]}(\vec{\mu}, \vec{\tau}, t; q)$$

$\vec{\mu}, \vec{\tau}$ are identified with fiber and base Kähler parameters.

So we have two totally different tools to compute $3d$ partition functions!

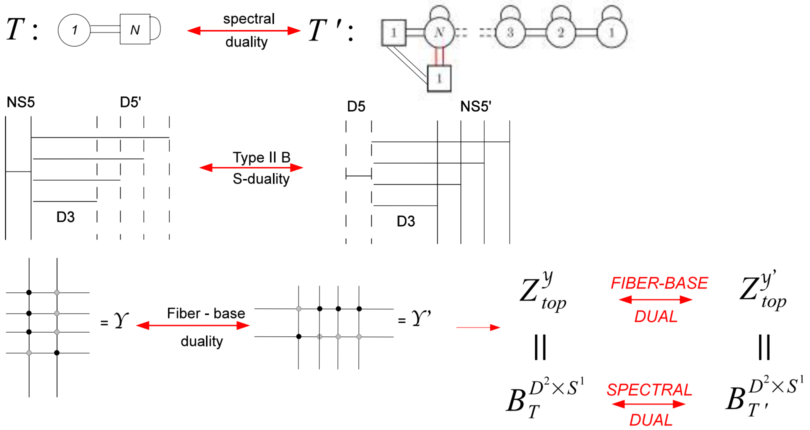
$3d$ duality from fiber-base duality

The CYs X, $\mathcal{X}$ (before and after Higgsing) are invariant under the action fiber-base or $5d$ S-duality which swaps μ_i with τ_i and so

$$
\begin{array}{ccc}
Z^{\mathcal{X}}_{top}(\vec{\mu}, \vec{\tau}; q, t) & \overset{\text{FIBER-BASE}}{\underset{\text{DUAL}}{\longleftrightarrow}} & Z^{\mathcal{X}'}_{top}(\vec{\tau}, \vec{\mu}; q, t) \\
\parallel & & \parallel \\
B^{D^2 \times S^1}_{FT[SU(N)]}(\vec{\mu}, \vec{\tau}, t; q) & \overset{\text{SPECTRAL}}{\underset{\text{DUAL}}{\longleftrightarrow}} & B^{D^2 \times S^1}_{FT[SU(N)]}(\vec{\tau}, \vec{\mu}, t; q)
\end{array}
$$

$\rightarrow$ 3d self-duality for $FT[SU(N)]$ descends from fiber-base duality.

More spectral dual pairs



We can generate many new 3d spectral dual pairs from fiber-base via Higgsing. [Aprile-SP-Zenkevich]

Reducing a spectral dual pair on $D^2 \times S^1$

- Higgs limit reduces $FT[SU(N)]$ theory to the same theory in $2d$:

$$\lim_{q\to 1} \mathcal{B}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\tau},\vec{\mu},t) = \mathcal{B}^{D_2}_{FT[SU(N)]}(\vec{\tau},\vec{f},\beta)\,.$$

- On the dual side the Higgs limit becomes a Coulomb limit yields an $N+2$ point DF free field correlator:

$$\lim_{q\to 1} \check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]}(\vec{\tau},\vec{\mu},t) = \langle\vec{\alpha}_\infty|\, V_{\vec{\alpha}_1}(z_1)\ldots V_{\vec{\alpha}_N}(z_N)\,\left(\prod_{a=1}^{N} Q^a_{(a)}\right)|\vec{\alpha}^{(0)}\rangle^{A_{N-1}}_{\text{free}}\,.$$

(screening charges↔ranks, insertion points↔masses, ...)

→ Reducing the 3d spectral dual pair yields Gauge/free field correspondence. We recover exactly the AGT map between the 2d GLSM describing the defect theory and the Toda correlator with degenerate operators [Gomis-Le Floch].

Gauge/q-CFT

$D^2 \times S^1$ partition functions of $\mathcal{N} = 2$ theories can be directly mapped to correlators of q-deformed Dotsenko-Fateev correlators
[Aganagic-Haouzi-Kozcaz-Shakirov], [Aganagic-Haouzi-Shakirov].

For $FT[SU(N)]$ then we have the following duality web:

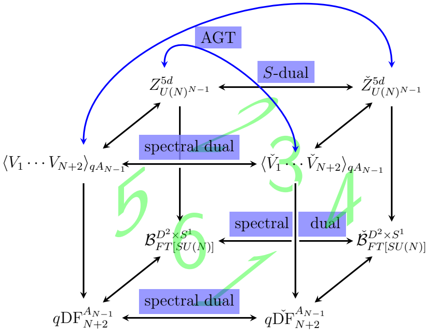
$$
\begin{array}{ccc}
\mathcal{B}^{D_2\times S^1}_{FT[SU(N)]} & \xleftrightarrow{\text{spectral dual}} & \check{\mathcal{B}}^{D_2\times S^1}_{FT[SU(N)]} \\
\text{gauge}/q\text{DF} \updownarrow & & \updownarrow \text{gauge}/q\text{DF} \\
q\text{DF}^{A_{N-1}}_{N+2} & \xleftrightarrow{\text{spectral dual}} & q\check{\text{DF}}^{A_{N-1}}_{N+2}
\end{array}
$$

- Horizontal arrows indicate (IR) dualities, requires highly non-trivial integral identities.
- Vertical arrows indicate AGT-like correspondenes. Trivial mapping, only need to establish a dictionary.

3d/q-CFT web via Higgsing

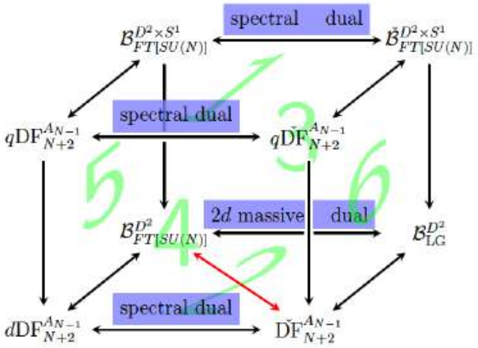
We saw that the $FT[SU(N)]$ spectral dual pair can be derived via Higgsing from 5d.

Similarly the $q\mathrm{DF}^{A_{N-1}}_{N+2}$ blocks can be obtained by tuning external and internal momenta in $\langle V_1 \cdots V_{N+2}\rangle_{q-A_{N-1}}$:



The AGT map corresponds to the blue arrows.

Reduction of the 3d/q-CFT web



- $\mathcal{B}^{D_2}_{LG}$ is the D^2 partition function of a theory of twisted chiral multiplets with a twisted superpotential.
- $dDF^{A_{N-1}}_{N+2}$ is a correlator of vertex operators with d-$\mathcal{W}_N$ symmetry, an *un-natural* limit of q-$\mathcal{W}_N$
- The red link is the familiar gauge/CFT correspondence connecting surface operators vortex partition functions to Toda degenerate correlators.

Let now see how starting from 3d Seiberg-like dualities we obtain duality identities between 2d free field correlators and what we can learn from this observation [SP-Sacchi I, II, '19].

3d IR Seiberg-like dualities

- Aharony duality:

  $\mathcal{T}$: $U(N_c)$ with N_f flav. $Q, \tilde{Q}$, $\mathcal{W} = 0$

  $\mathcal{T}'$: $U(N_f - N_c)$ with N_f flav. $q, \tilde{q}$, $\mathcal{W} = S_-\hat{\mathfrak{M}}^+ + S_+\hat{\mathfrak{M}}^- + Mq\tilde{q}$

- Monopole duality I: [Benini-Benvenuti-SP]

  $\mathcal{T}_{\mathfrak{M}}$: $U(N_c)$ with N_f flav. $Q, \tilde{Q}$, $\mathcal{W} = \mathfrak{M}^+$

  $\mathcal{T}'_{\mathfrak{M}}$: $U(N_f - N_c - 1)$ with N_f flav. $q, \tilde{q}$, $\mathcal{W} = \hat{\mathfrak{M}}^- + S_+\hat{\mathfrak{M}}^+ + Mq\tilde{q}$.

- Monopole duality II: [Benini-Benvenuti-SP]

  $\mathcal{T}_{\mathfrak{M}}$: $U(N_c)$ with N_f flav. $Q, \tilde{Q}$, $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$

  $\mathcal{T}'_{\mathfrak{M}}$: $U(N_f - N_c - 2)$ with N_f flavors $q, \tilde{q}$, $\mathcal{W} = \hat{\mathfrak{M}}^+ + \hat{\mathfrak{M}}^- + Mq\tilde{q}$

From 3d dualities to DF duality relations

For example, by carefully taking the Coulomb limit of the Monopole duality II on $S^2 \times S^1$ we land on an identity between complex integrals found by [Fateev-Litvinov]:

$$\int \prod_{i=1}^{N_c} d^2 z_i \prod_{i<j}^{N_c} |z_i - z_j|^2 \prod_{i=1}^{N_c} \prod_{a=1}^{N_f} |z_i - t_a|^{2p_a} = \prod_{a=1}^{N_f} \gamma(1+p_a) \prod_{a<b}^{N_f} |t_a - t_b|^{2(1+p_a+p_b)} \times$$

$$\times \int \prod_{i=1}^{N_f - N_c - 2} d^2 z_i \prod_{i<j}^{N_f - N_c - 2} |z_i - z_j|^2 \prod_{i=1}^{N_f - N_c - 2} \prod_{a=1}^{N_f} |z_i - t_a|^{-2(1+p_a)} ,$$

with the condition $\sum_{a=1}^{N_f} p_a = -N_c - 1$ and

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} ,$$

This identity and its various specialization are the so called duality relations for the Dotesenko-Fateev integrals appearing in various CFT context.

Once upon a time

[Goulian-Li] observed that in Liovulle CFT (a 2d boson with $e^{b\phi}$ interaction) correlator exhibit poles. For example:

$$\underset{\alpha_1+\alpha_2+\alpha_3=Q-Nb}{\text{res}} \langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle = (-\pi\mu)^N I_N(\alpha_1,\alpha_2,\alpha_3)$$

at these values the correlator is represented by Dotsenko-Fateev free field correlator with N screening charges:

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)(Q)^N\rangle_{free}\,, \qquad Q=\int dx\; e^{b\phi}$$

expanding the free field in oscillators and normal-ordering we are left with the evaluation of a complex integral:

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \int \prod_{k=1}^{N} d^2\vec{t}_k\; |t_k|^{-4b\alpha_1}|t_k-1|^{-4b\alpha_2}\prod_{i<j}^{N}|t_i-t_j|^{-4b}\,.$$

To determine $\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle$ for generic momenta lifting the screening constraint we need to be able to do analytic continuation in N.

To evaluate the integral one can use the *duality relations* between complex DF integrals and find a recursion

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \frac{\gamma(-Nb^2)}{\gamma(-b^2)} \frac{1}{\gamma(2b\alpha_1)\gamma(2b\alpha_2)\gamma(2b\alpha_3+(N-1)b^2)}\; I_{N-1}(\alpha_1+b/2,\alpha_2+b/2,\alpha_3).$$

Repeating this procedure N times, we obtain the evaluation formula:

$$I_N(\alpha_1,\alpha_2,\alpha_3) = \prod_{k=1}^{N}\left(\frac{\gamma(-kb^2)}{\gamma(-b^2)}\right)\prod_{j=0}^{N-1}\frac{1}{\gamma(2b\alpha_1+jb^2)\gamma(2b\alpha_2+jb^2)\gamma(2b\alpha_3+jb^2)}.$$

We can rewrite I_N in a form parametric in N which allows for analytic continuation, lifting the screening condition:

$$\langle V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle = C(\alpha_1,\alpha_2,\alpha_3) \sim \frac{\Upsilon'(0)\prod_{k=1}^{3}\Upsilon(2\alpha_k)}{\Upsilon(\alpha-Q)\prod_{k=1}^{3}\Upsilon(\alpha-2\alpha_k)},$$

where $\Upsilon(x)$ satisfies the functional relations

$$\Upsilon(x+b) = \gamma(bx)b^{1-2bx}\Upsilon(x) \qquad \text{same for } b\to b^{-1}\,.$$

This is the celebrated Dorn-Otto-Zamolodchikov-Zamolodchikov formula.

3d dualities *from* 2d DF duality relations?

We saw that 3d Seiberg-like dualities reduce to the DF duality relations appearing in the derivation of the Liouville 3-point function.

Can we reverse the logic? Can we *uplift* to a genuine 3d duality also the evaluation formula for the I_N integral?

Yes!

The *uplift* we are looking is a recently proposed 3d duality [Benvenuti]:

- **Theory A**: $U(N)$ with adjoint Φ, one flavor P, $\tilde{P}$, N singlets β_j:

$$\mathcal{W} = \sum_{j=1}^{N} \beta_j \mathrm{Tr}\Phi^j \,.$$

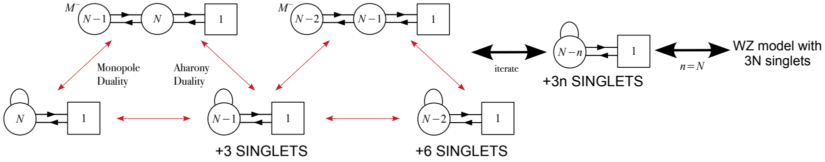
- **Theory B**: Wess-Zumino model with $3N$ singlets α_j, $T_j^{\pm}$:

$$\hat{\mathcal{W}} = \sum_{i,j,l=1}^{N} \alpha_i \, T_j^+ \, T_{N-l+1}^- \delta_{i+j+l,2N+1} \,.$$

Global symmetry: $U(1)_\tau \times U(1)_\zeta \times U(1)_\mu$. Operator map:

$$\begin{array}{rcll}
\textbf{Theory A} & \leftrightarrow & \textbf{Theory B} & \\
\mathfrak{M}^+_{\Phi^s} & \leftrightarrow & T^+_{s+1} & \\
\mathfrak{M}^-_{\Phi^s} & \leftrightarrow & T^-_{N-s} & \\
\mathrm{Tr}\left(\tilde{P}\Phi^s P\right) & \leftrightarrow & \alpha_{s+1}, & s = 0, \cdots, N-1 \,.
\end{array}$$

We can prove this duality by iterating 3d basic dualities:



Of course localised partition functions on any 3-manifold match. For example on the 3-sphere:

$$\mathcal{Z}^{S^3}_{U(N)} = \int \prod_{a=1}^{N} dx_a e^{2\pi i \zeta \sum_a x_a} \frac{\prod_{a,b=1}^{N} S_2\left(Q - i(x_a - x_b) + 2i\tau\right)}{\prod_{a<b}^{N} S_2\left(Q \pm i(x_a - x_b)\right)} S_2\left(Q \pm i x_a + i\mu\right) =$$

$$= \prod_{j=1}^{N} S_2\left(Q + 2ij\tau\right) S_2\left(Q + 2i\mu + 2i(j-1)\tau\right) \times$$

$$\times S_2\left(\frac{Q}{2} + i\zeta - i\mu - 2i(N-j)\tau\right) S_2\left(\frac{Q}{2} - i\zeta - i\mu - 2i(j-1)\tau\right) = \mathcal{Z}^{S^3}_{WZ} \,,$$

where $S_2(x|\omega_1, \omega_2) \equiv S_2(x)$ is the function appearing in 1-loop computations on a 3-sphere $\omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 = 1$.

So we succeeded in *uplifting* the evaluation formula for the 3-point correlator to a genuine 3d IR duality.

What about the analytic continuation? can we make sense of it?

YES!

Analytic continuation as geometric transition

The function $S_3(z|\omega_1, \omega_2, \omega_3)$ appears in 1-loop computations on a 5-sphere $\omega_1^2|z_1|^2 + \omega_2^2|z_2|^2 + \omega_3^2|z_3|^2 = 1$ and satisfies:

$$S_3(z + \omega_3|\omega_1, \omega_2, \omega_3) = S_2(z|\omega_1, \omega_2) S_3(z|\omega_1, \omega_2, \omega_3)$$

which allows us to rewrite the 3d WZ partition function as

$$\mathcal{Z}_{WZ}^{S^3} = \operatorname*{Res}_{N\in\mathbb{N}} \frac{S_3'(0)\, S_3(-2i\mu + 2i\tau)\, S_3\left(\frac{Q}{2} \pm i\zeta - i\mu - 2i(N-1)\tau\right)}{S_3(-2iN\tau)\, S_3(-2i\mu)\, S_3\left(\frac{Q}{2} \pm i\zeta - i\mu + 2i\tau\right)} = \operatorname*{Res}_{N\in\mathbb{N}} \mathcal{Z}_{T_2}^{S^5}.$$

On the RHS we recognise the partition function of the 5d T_2 theory!

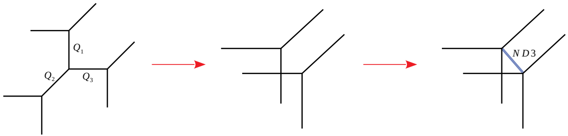
We identify $\omega_3 = 2i\tau$ while μ, ζ and $N\tau$ with the fugacities for the global symmetry $SU(2)^3$ of T_2.

How do we interpret the quantization condition of the fugacity in the 5d theory?

Analytic continuation as geometric transition

The $5d$ T_N theory can be realised on a web of intersecting $(0,1)$-branes, $(1,0)$-branes and $(1,1)$-branes [Benini-Benvenuti-Tachikawa]:

Equivalently we can geometrically engineer T_N by M-theory on the toric Calabi-Yau tree-fold $\frac{\mathbb{C}^3}{\mathbb{Z}_N \times \mathbb{Z}_N}$ with toric diagram given by the brane web.



The quantization condition on the parameters implies that one Kähler parameter is quantized $Q = q^N q^{1/2} t^{-1/2}$. This leads to geometric transition.

We arrive at a configuration of N D3 branes stretched between two pieces of the web. The defect theory living on the N D3 branes is our $3d$ $U(N)$ theory with one adjoin and one flavor.

This curious connection between the 3-point Liouville correlator and the 3d $U(N)$ theory is just the starting point!

We already have many more examples of 3d dualities *from* free field correlators.

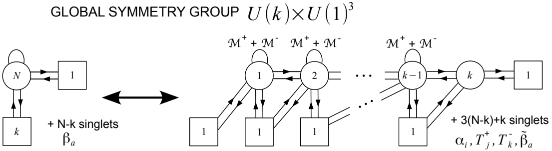
For example the $k+3$-point free field correlator.

$$\underset{\alpha_1+\alpha_2+\alpha_3=Q-\frac{k}{2}-Nb}{\mathrm{res}} \langle V_{-\frac{b}{2}}(z_1)\cdots V_{-\frac{b}{2}}(z_k)V_{\alpha_1}(0)V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle = (-\pi\mu)^N I_N^k(\alpha_1,\alpha_2,\alpha_3)$$

the rank N integral can be massaged in a form suitable for analytic continuation using the Kernel function:

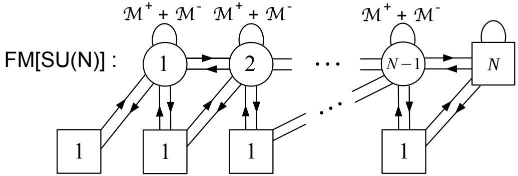
$$I_N^k(\alpha_1,\alpha_2,\alpha_3) \sim \int \prod_{i=1}^{k} d^2\vec{x}_i \prod_{i<j}^{k} |x_i-x_j|^{-4b}|x_i|^{2A}|x_i-1|^{2B} K_k^C(x_1,..,x_k|z_1,..,z_k)\,,$$

This duality identity can be uplifted to a very non-trivial 3d IR duality!



On the rhs N enters only parametrically inside the R-charges and in the number of singlets (which can be expressed as $5d$ hypers as in the $k=0$).

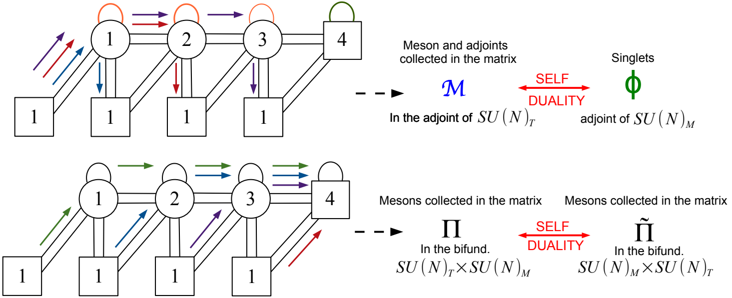
$FM[SU(N)]$: the $3d$ avatar of the kernel function



- Invariant under a duality swapping $SU(N)_T \leftrightarrow SU(N)_M$
- Reduces to $FT[SU(N)]$ when real mass for $U(1)_\Delta$ is turned on
- The $S^2 \times S^1$ partition function of $FM[SU(N)]$ reduces in the 2d Coulomb limit to $K_N^C(x_1, .., x_N | z_1, .., z_N)$

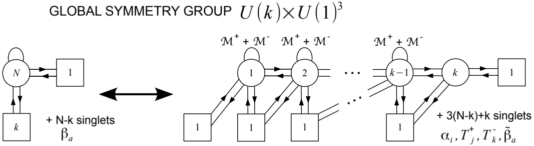
Self-duality tests

- Operator map



- Explicit map of partition functions (iterating fundamental identities) up tp $N = 2$.
- Matching various orders of the R-symmetry fuguacity expansion of the $S^2 \times S^1$ index, up to $N = 4$.

Rank-stabilization duality



$$
\begin{aligned}
\mathrm{Tr}_N \Phi^{N-k+a} \quad &\leftrightarrow \quad \tilde{\beta}_a, \qquad a = 1, \cdots, k \\
\mathfrak{M}^+_{\Phi^s} \quad &\leftrightarrow \quad \begin{cases} T^+_{s+1} & s = 0, \cdots, N-k-1 \\ \mathfrak{M}^+_{\mathbb{M}^{k-N+s}} & s = N-k, \cdots, N \end{cases} \\
\mathfrak{M}^-_{\Phi^s} \quad &\leftrightarrow \quad \begin{cases} T^-_{N-s} & s = 0, \cdots, N-k-1 \\ \mathfrak{M}^-_{\mathbb{M}^{k-N+s}} & s = N-k, \cdots, N \end{cases} \\
\mathrm{Tr}_N \left(\tilde{P}\Phi^s P\right) \quad &\leftrightarrow \quad \begin{cases} \alpha_{s+1} & s = 0, \cdots, N-k-1 \\ \mathrm{Tr}_k \left(\tilde{p}\mathbb{M}^{k-N+s} p\right) & s = N-k, \cdots, N-1 \end{cases} \\
Q\tilde{P} \quad &\leftrightarrow \quad \tilde{\Omega} \\
P\tilde{Q} \quad &\leftrightarrow \quad \Omega \\
Q\tilde{Q} \quad &\leftrightarrow \quad \mathcal{M}\,.
\end{aligned}
$$

Tests: partition functions match (iterating fundamental identities) up to $k = 2$ and perturbative match of the index up to $k = 3$.

Outlook

- The $FM[SU(N)]$ theory is self-dual and reduces to $FT[SU(N)]$ in a real mass deformation
- Can be used as building block to construct more general theories.
- Is $FM[SU(N)]$ a duality wall?
- $FM[SU(N)]$ is related to the Kernel function for q-deformed hypergeometric functions of N variables and to Macdonald polynomials. These objects have an elliptic analogue, is there a $4d$ uplift?

THANK YOU!