

\mathcal{W} -symmetry and instanton \mathcal{R} -matrix

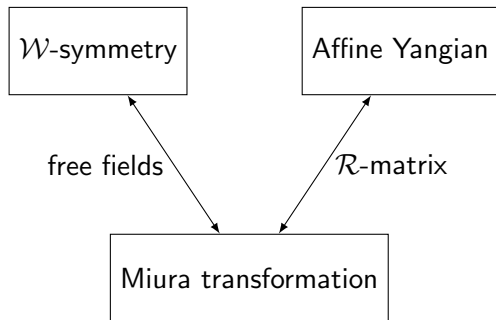
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Plan

'algebraic structures in 2d conformal field theory with higher spins'



\mathcal{W}_∞ : (spherical) degenerate double affine Hecke algebra (Cherednik), cohomological Hall algebra (Kontsevich & Soibelman), Yangian of $\widehat{\mathfrak{gl}(1)}$, quantum toroidal algebra, Ding-Iohara-Miki algebra

W algebras

\mathcal{W} -algebras: extensions of the Virasoro algebra (2d CFT) by higher spin currents - appear in many different contexts:

- integrable hierarchies of PDE (KdV/KP) \rightsquigarrow \mathcal{W} is quant. KP
- (old) matrix models
- instanton partition functions and AGT
- holographic dual description of 3d higher spin theories
- quantum Hall effect
- topological strings
- higher spin square (Gaberdiel, Gopakumar)
- $\mathcal{N} = 4$ SYM at junction of three codimension 1 defects (Gaiotto, Rapčák)
- geometric representation theory (equivariant cohomology of various moduli spaces)

Zamolodchikov \mathcal{W}_3 algebra

\mathcal{W}_3 algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.}$$

together with spin 3 primary field $W(w)$

$$T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + \text{reg.}$$

To close the algebra we need to find the OPE of W with itself consistent with associativity (Jacobi, crossing symmetry...).

The result:

$$\begin{aligned}
 W(z)W(w) \sim & \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
 & + \frac{1}{(z-w)^2} \left(\frac{32}{5c+22} \Lambda(w) + \frac{3}{10} \partial^2 T(w) \right) \\
 & + \frac{1}{z-w} \left(\frac{16}{5c+22} \partial \Lambda(w) + \frac{1}{15} \partial^3 T(w) \right) + \text{reg.}
 \end{aligned}$$

Λ is a quasiprimary 'composite' (spin 4) field,

$$\Lambda(z) = (TT)(z) - \frac{3}{10} \partial^2 T(z).$$

The algebra is non-linear, not a Lie algebra in the usual sense (linearity should not be expected for spins ≥ 3 or non-abelian ≥ 2).

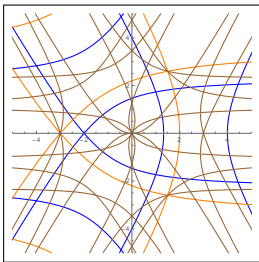
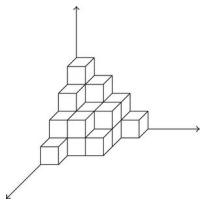
- \mathcal{W}_∞ : generalizing \mathcal{W}_3 to fields of spin $2, 3, 4, 5, \dots$
- solving associativity conditions for this field content \rightsquigarrow two-parameter family (Gaberdiel-Gopakumar)
- parameters: central charge c and rank parameter λ
- choosing $\lambda = N \rightarrow$ truncation of \mathcal{W}_∞ to $\mathcal{W}_N = \mathcal{W}[\mathfrak{sl}(N)]$, i.e. \mathcal{W}_∞ is interpolating algebra for the whole A series of \mathcal{W}_N algebras (cf. $\mathfrak{sl}(N)$ vs $hs[\lambda]$ in $3d$ higher spin gravities)
- adding spin 1 field, we have $\mathcal{W}_{1+\infty} \rightsquigarrow$ many simplifications
- surprise: $\mathcal{W}_{1+\infty}$ contains all $\mathcal{W}[\mathfrak{g}]$ for all simple \mathfrak{g} (except possibly \mathfrak{f}_4 ?)

- triality symmetry of the algebra (Gaberdiel & Gopakumar)

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \quad c = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)$$

- MacMahon function as vacuum character of the algebra

$$\prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \dots$$



The resulting OPEs look like

$$\begin{aligned}
 U_3(z)U_5(w) &\sim \frac{1}{(z-w)^7} \left(\frac{1}{2} \alpha(n-3)(n-2)(n-1)n \left(4\alpha^2 \left(\alpha^2(n(5n-9)+1) - 3n+4 \right) + 1 \right) \mathbb{1} \right) \\
 &+ \frac{1}{(z-w)^6} \left(\frac{1}{6} (n-3)(n-2)(n-1) \left(6\alpha^4 n(2n-3) + \alpha^2(10-9n) + 1 \right) U_1(w) \right) \\
 &+ \frac{1}{(z-w)^5} \left(-\alpha(n-3)(n-2)(n-1) \left(-4\alpha^2 + 3\alpha^2 n - 1 \right) (U_1 U_1)(w) \right) \\
 &+ \alpha(n-3)(n-2) \left(4\alpha^2 n^2 - 4\alpha^2 n - n - 2 \right) U_2(w) \\
 &- \frac{1}{2} \alpha^2(n-3)(n-2)(n-1) \left(4\alpha^2 n(2n-3) - 3n+2 \right) U_1'(w) \\
 &+ \frac{1}{(z-w)^4} \left(-\alpha(n-3)(n-2)(n-1) \left(\alpha^2(3n-4) - 1 \right) (U_1' U_1)(w) \right) \\
 &- \frac{1}{2} (n-3)(n-2) \left(2\alpha^2(n-1) - 1 \right) (U_1 U_2)(w) \\
 &+ (n-3) \left(\alpha^2(n^2+2) - 3 \right) U_3(w) \\
 &- \frac{1}{4} \alpha^2(n-3)(n-2)(n-1) \left(4\alpha^2 n(2n-3) - 3n+2 \right) U_1''(w) \\
 &+ \alpha(n-3)(n-2) \left(\alpha^2(n-1)n - 1 \right) U_2'(w) \\
 &+ \dots
 \end{aligned}$$

But there are special properties of OPE that allow guessing a general formula

- a basis of local fields $U_j(z)$ in which the OPEs have purely quadratic non-linearity (impossible in primary basis)
- all the derivative terms can be resummed and we find a bilocal expansion (Lukyanov)

$$U_j(z)U_k(w) + \sum_{l+m < j+k} C_{jk}^{lm}(N, \alpha_0) \frac{U_l(z)U_m(w)}{(z-w)^{j+k-l-m}} \sim \text{reg.}$$

- other special properties like $C_{jk}^{lm}(N) = C_{j+1k+1}^{l+1m+1}(N+1)$
- how to explain these?

Yangian of $\widehat{\mathfrak{gl}(1)}$

The Yangian of $\widehat{\mathfrak{gl}(1)}$ (Arbesfeld-Schiffmann-Tsymbaliuk) is an associative algebra with generators $\psi_j, e_j, f_j, j \geq 0$ and relations

$$0 = [e_{j+3}, e_k] - 3[e_{j+2}, e_{k+1}] + 3[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\ + \sigma_2 [e_{j+1}, e_k] - \sigma_2 [e_j, e_{k+1}] - \sigma_3 \{e_j, e_k\}$$

$$0 = [f_{j+3}, f_k] - 3[f_{j+2}, f_{k+1}] + 3[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}] \\ + \sigma_2 [f_{j+1}, f_k] - \sigma_2 [f_j, f_{k+1}] + \sigma_3 \{f_j, f_k\}$$

$$0 = [\psi_{j+3}, e_k] - 3[\psi_{j+2}, e_{k+1}] + 3[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\ + \sigma_2 [\psi_{j+1}, e_k] - \sigma_2 [\psi_j, e_{k+1}] - \sigma_3 \{\psi_j, e_k\}$$

$$0 = [\psi_{j+3}, f_k] - 3[\psi_{j+2}, f_{k+1}] + 3[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}] \\ + \sigma_2 [\psi_{j+1}, f_k] - \sigma_2 [\psi_j, f_{k+1}] + \sigma_3 \{\psi_j, f_k\}$$

$$0 = [\psi_j, \psi_k]$$

$$\psi_{j+k} = [e_j, f_k]$$

'initial/boundary conditions'

$$\begin{aligned} [\psi_0, e_j] &= 0, & [\psi_1, e_j] &= 0, & [\psi_2, e_j] &= 2e_j, \\ [\psi_0, f_j] &= 0, & [\psi_1, f_j] &= 0, & [\psi_2, f_j] &= -2f_j \end{aligned}$$

and finally the Serre relations

$$0 = \text{Sym}_{(j_1, j_2, j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]], \quad 0 = \text{Sym}_{(j_1, j_2, j_3)} [f_{j_1}, [f_{j_2}, f_{j_3+1}]].$$

There are 3 parameters of the algebra, $h_1, h_2, h_3 \in \mathbb{C}$ constrained by $h_1 + h_2 + h_3 = 0$. Furthermore, we used the shorthand notation

$$\begin{aligned} \sigma_2 &= h_1 h_2 + h_1 h_3 + h_2 h_3 \\ \sigma_3 &= h_1 h_2 h_3. \end{aligned}$$

We have both commutators and anticommutators in defining quadratic relations (but no \mathbb{Z}_2 grading) - for $\sigma_3 \neq 0$ not a Lie (super)-algebra.

Introducing generating functions

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \quad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \quad \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$

the first set of formulas above (almost!!) simplify to

$$\begin{aligned} e(u)e(v) &= \varphi(u-v)e(v)e(u), & f(u)f(v) &= \varphi(v-u)f(v)f(u), \\ \psi(u)e(v) &= \varphi(u-v)e(v)\psi(u), & \psi(u)f(v) &= \varphi(v-u)f(v)\psi(u) \end{aligned}$$

with rational structure function

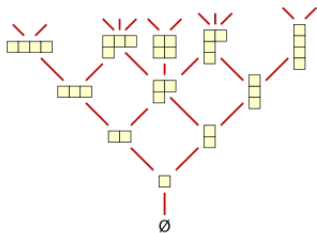
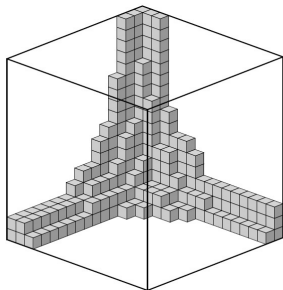
$$\varphi(u) = \frac{(u+h_1)(u+h_2)(u+h_3)}{(u-h_1)(u-h_2)(u-h_3)}$$

The representation theory of the algebra is much simpler in this Yangian formulation and it is controlled by this simple function (looks like a 3d generalization of harmonic oscillator)

$\psi(u)$, $e(u)$ and $f(u)$ in representations act like

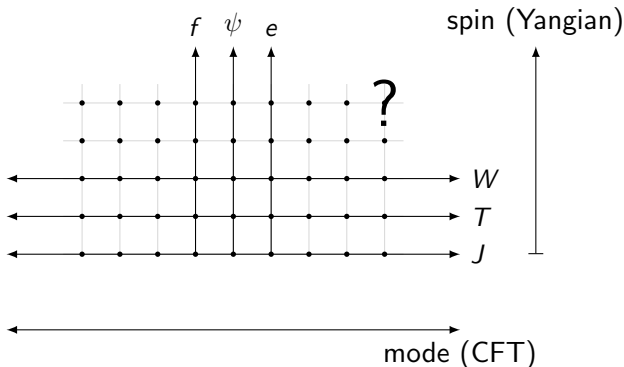
$$\psi(u) |\Lambda\rangle = \psi_0(u) \prod_{\square \in \Lambda} \varphi(u - h_{\square}) |\Lambda\rangle, \quad e(u) |\Lambda\rangle = \sum_{\square \in \Lambda^+} \frac{E(\Lambda \rightarrow \Lambda + \square)}{u - h_{\square}} |\Lambda + \square\rangle$$

where the states $|\Lambda\rangle$ are associated to geometric configurations of boxes (plane partitions, ...) and where $h_{\square} = \sum_j h_j x_j(\square)$ is the weighted geometric position of the box.



Two different descriptions of the algebra:

- usual CFT point of view with local fields $J(z)$, $T(z)$, $W(z)$, ... with increasingly complicated OPE as we go to higher spins
- Yangian point of view (Arbesfeld-Schiffmann-Tsymbaliuk) where all the spins are included in the generating functions $\psi(u)$, $e(u)$ and $f(u)$ but accessing higher mode numbers is difficult



How to connect these two descriptions?

- the parameters can be identified as $\lambda_1 = -\psi_0 h_2 h_3$
- the generators with low spin and mode can be identified

$$\psi_2 = 2L_0, \quad e_0 = J_{-1}, \quad f_0 = -J_{+1}$$

$$\psi_3 = (W_3 + \dots)_0 + \sigma_3 \sum_{m>0} (3m-1) J_{-m} J_m$$

(cut & join operator, not a zero mode of a local field!)

- this is sufficient to find the map in principle, but what is the more conceptual way to understand the map?
- Negut: closed form of the map $\mathcal{W} \rightarrow \mathcal{Y}$ using shuffle algebra

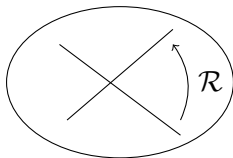
Miura transformation and \mathcal{R} -matrix

- Miura transformation - free field representation of \mathcal{W}_N

$$(\partial + \partial\phi_1(z)) \cdots (\partial + \partial\phi_N(z)) = \sum_{j=0}^N U_j(z) \partial^{N-j}$$

- quantization of differential operators
- the embedding of \mathcal{W}_N in the bosonic Fock space depends on the way we order the fields
- Maulik-Okounkov: \mathcal{R} -matrix as transformation (intertwiner) between two embeddings, $\mathcal{R} : \mathcal{F}^{\otimes 2} \rightarrow \mathcal{F}^{\otimes 2}$

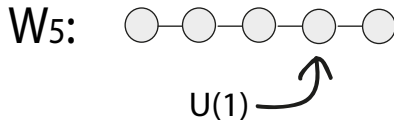
$$(\partial + \partial\phi_1)(\partial + \partial\phi_2) = \mathcal{R}^{-1}(\partial + \partial\phi_2)(\partial + \partial\phi_1)\mathcal{R}$$



- \mathcal{R} defined in this way satisfies the Yang-Baxter equation (two ways of reordering $321 \rightarrow 123$)

$$\begin{aligned} \mathcal{R}_{12}(u_1 - u_2)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{23}(u_2 - u_3) &= \\ &= \mathcal{R}_{23}(u_2 - u_3)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{12}(u_1 - u_2) \end{aligned}$$

- the spectral parameter u - the global $u(1)$ charge
- \mathcal{R} -matrix satisfying YBE \rightsquigarrow apply the algebraic Bethe ansatz



- spin chain of length $N \rightsquigarrow \mathcal{W}_N$ algebra

- consider an *auxiliary* Fock space \mathcal{F}_A and a *quantum* space $\mathcal{F}_Q \equiv \mathcal{F}_1 \otimes \dots \otimes \mathcal{F}_N$
- we associate to this the monodromy matrix $\mathcal{T}_{AQ} : \mathcal{F}_A \otimes \mathcal{F}_Q \rightarrow \mathcal{F}_A \otimes \mathcal{F}_Q$ defined as

$$\mathcal{T}_{AQ} = \mathcal{R}_{A1}\mathcal{R}_{A2}\cdots\mathcal{R}_{AN}$$

- if the individual \mathcal{R} -matrices satisfied the YBE, \mathcal{T} will also satisfy YBE with respect to two auxiliary spaces A and B

$$\mathcal{R}_{AB}\mathcal{T}_A\mathcal{T}_B = \mathcal{T}_B\mathcal{T}_A\mathcal{R}_{AB}$$

- the algebra of matrix elements of \mathcal{T} satisfying this equation is the Yangian (quadratic defining relations!) - RTT presentation

- in our situation the Fock-Fock \mathcal{R} -matrix controlling the commutation relations of Yangian has rows and columns labeled by Young diagrams
- we can however restrict to simple matrix elements, i.e. upper left corner of the \mathcal{R} -matrix

$$\mathcal{H} = \langle 0|_A \mathcal{T} |0\rangle_A, \quad \mathcal{E} = \langle 0|_A \mathcal{T} |1\rangle_A, \quad \mathcal{F} = \langle 1|_A \mathcal{T} |0\rangle_A$$

- the YBE now implies relations between these operators like

$$0 = [\mathcal{H}(u), \mathcal{H}(v)]$$

(infinite set of commuting Hamiltonians) or

$$(u - v + h_3)\mathcal{H}(u)\mathcal{E}(v) = (u - v)\mathcal{E}(v)\mathcal{H}(u) + h_3\mathcal{H}(v)\mathcal{E}(u)$$

(ladder operators)

- how are these generating functions related to AST?

$$\psi(u) = \frac{u + \sigma_3 \psi_0}{u} \frac{\mathcal{H}(u + h_1) \mathcal{H}(u + h_2)}{\mathcal{H}(u) \mathcal{H}(u + h_1 + h_2)}$$

$$e(u) = h_3^{-1} \mathcal{H}(u)^{-1} \mathcal{E}(u), \quad f(u) = -h_3^{-1} \mathcal{F}(u) \mathcal{H}(u)^{-1}$$

- using \mathcal{R} -matrix one can find these Yangian generators systematically following the algorithm of QISM (the only input is the Miura transformation) \rightsquigarrow generalization
- verification of the relations satisfied by these is difficult and MO Yangian seems in fact to be bigger than AST Yangian (Schiffmann: they differ by central elements)

How does the \mathcal{R} -matrix look like?

- a fermionic form obtained by A. Smirnov by studying the large N limit of $\mathfrak{gl}(N)$ \mathcal{R} -matrix - complicated
- a bosonic formula is not known so far, but results of Nazarov-Sklyanin can be interpreted as matrix elements of \mathcal{R} -matrix in mixed representation where the two bosonic Fock spaces are associated to different asymptotic directions of \mathcal{W}_∞

$$\begin{aligned} \mathcal{R}(u) = & \mathbb{1} - \frac{1}{u} \sum_{j>0} a_{-j} a_j + \frac{1}{2!u(u+1)} \sum_{j,k>0} (a_{-j} a_{-k} + a_{-j-k}) (a_j a_k + a_{j+k}) \\ & - \frac{1}{3!u(u+1)(u+2)} \sum_{j,k,l>0} (a_{-j} a_{-k} a_{-l} + a_{-j-k} a_{-l} + a_{-j-l} a_{-k} \\ & \qquad \qquad \qquad + a_{-k-l} a_{-j} + 2a_{-j-k-l}) (\dots) + \dots \end{aligned}$$

with $a_j \equiv a_j^{(1)} - a_j^{(2)}$.

Application: Calogero models

- instead of building the monodromy matrix \mathcal{T} from the Fock-Fock \mathcal{R} -matrices, we could use the Miura operators $\partial + \partial\phi(z)$, using for the quantum space the space of functions of z coordinate
- this produces for free commuting Hamiltonians of quantum Calogero models as well as ladder operators

$$\mathcal{H}(u) = \sum_{\text{pairings}} \prod_{\text{pairs } j < k} \left(-\frac{h_1 h_2}{u^2} \frac{1}{4 \sinh^2 \left(\frac{z_j - z_k}{2} \right)} \right) \prod_{\ell \text{ unpaired}} \left(1 + u^{-1} h_3 \partial_{z_\ell} \right)$$

(Feynman, Polychronakos)

- moreover these are guaranteed to satisfy the same commutation relations $\rightsquigarrow \mathcal{W}_\infty$ action on orbital DOF
- the Hamiltonians can be easily generalized to elliptic case, what about ladder operators?

Lessons, questions

- non-linearity is to be expected and probably cannot be avoided (but quadraticity? Miura, Yangian, RTT)
- other algebras with similarly nice properties? Grassmannian \mathcal{W} ? higher spin square?
- Yangian description of the orthosymplectic case? exceptional subalgebras?
- many things can be generalized to include matrix DOF (rectangular \mathcal{W} , spin Calogero models - $\mathcal{W}_{1+\infty}[GL(M)]$)
- growth of states like of 3d theory, global subalgebra diffeomorphisms of S^2 , any connection to membrane?
- classical KP hierarchy has very well developed theory, what are the translations of objects there to the quantum language? tau functions, Hirota equations, Y-systems, T-systems etc.
- what are mirror symmetry, spectral curves, wall-crossing, topological recursion in this CFT language?

Thank you!

